family sizes from data which is limited to the family order of birth of a group of children. The treatment required by each problem owes its individuality only to the fact that the method of application may be designed to simplify the computation. Such simplification is attained by a suitable choice of class boundaries for the two variates.

## ON THE DENSEST PACKING OF CIRCLES

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**1. Introduction.** We show in this note that at most $A/\sqrt{12}$ circles of radius 1 can be placed in a convex polygon of area $A$ and with angles not greater than $\frac{2}{3}\pi$,* such that no two of the circles overlap. This upper bound for the number of inscribed circles is the best possible one, but the restriction on the angles is essential. Somewhat similar results have been obtained, using an entirely different method, by A. Thue.† More recently, the problem was studied by L. Fejes‡ and R. Rado.§

Apart from a simple application of differential calculus, our method is elementary. The proof is based on two lemmas which have a certain interest in themselves.

**2. The convex polygon $S(P)$.** Let $\Sigma$ be a set of points $P, P_1, P_2, \cdots$ in the plane $\pi$, such that the distance $\overline{P'P''}$ of any two different points $P', P''$ of $\Sigma$ is at least 2. Hence, if $C(P), C(P_1), C(P_2), \cdots$ are the circles of radius 1 and centers $P, P_1, P_2, \cdots$, then no two of these circles overlap.

Let $K$ be any circle in $\Pi$, say of radius $r$, and let $K'$ be the concentric circle of radius $r+1$. If now $P', P'', \cdots, P^{(l)}$ are any points of $\Sigma$ in $K$, then the circles $C(P'), C(P''), \cdots, C(P^{(l)})$ are contained in $K'$; their total area $l\pi$ is therefore not larger than the area $(r+1)^2\pi$ of $K'$, and so $l \leqq (r+1)^2$. Hence *every circle contains at most a finite number of points of $\Sigma$.*

For every point $P$ of $\Sigma$, denote by $S(P)$ the set of all points $Q$ in $\Pi$ for which

$$\overline{PQ} \leqq \overline{P_kQ} \qquad (k = 1, 2, \cdots). \tag{1}$$

The circle $C(P)$ is a subset of $S(P)$. For, if $Q$ lies in $C(P)$, then

$$\overline{P_kQ} \geqq \overline{PP_k} - \overline{PQ} \geqq 2 - 1 = 1 \geqq \overline{PQ}, \quad \text{since} \quad \overline{PQ} \leqq 1,\ \overline{PP_k} \geqq 2.$$

Denote by $\Lambda_k$ the locus of all points $Q$ for which $\overline{PQ} = \overline{P_kQ}$. Evidently $\Lambda_k$ is the line perpendicular to the line $PP_k$ which intersects the segment $PP_k$ at

* All angles in this paper are measured in radians. Moreover, we use the same symbols for the angles and their measures in radians.

† See his note read at the Scandinavian Mathematical Congress of 1892, and the paper, Norske Vid. Selsk. Skr. 1910, No. 1.

‡ Math. Z. 46, 1940, 83–85.

§ Dr. Rado was so kind as to give us the reference to Fejes's note, and he also informed us of his, as yet unpublished, results.

its center, say $P^{(k)}$. The points $Q$ satisfying $\overline{PQ} \leqq \overline{P_kQ}$ form the semi-plane determined by $\Lambda_k$ containing $P$, and this is a convex region. Therefore, from (1), $S(P)$ *is a closed convex region.* (Fig. 1.)

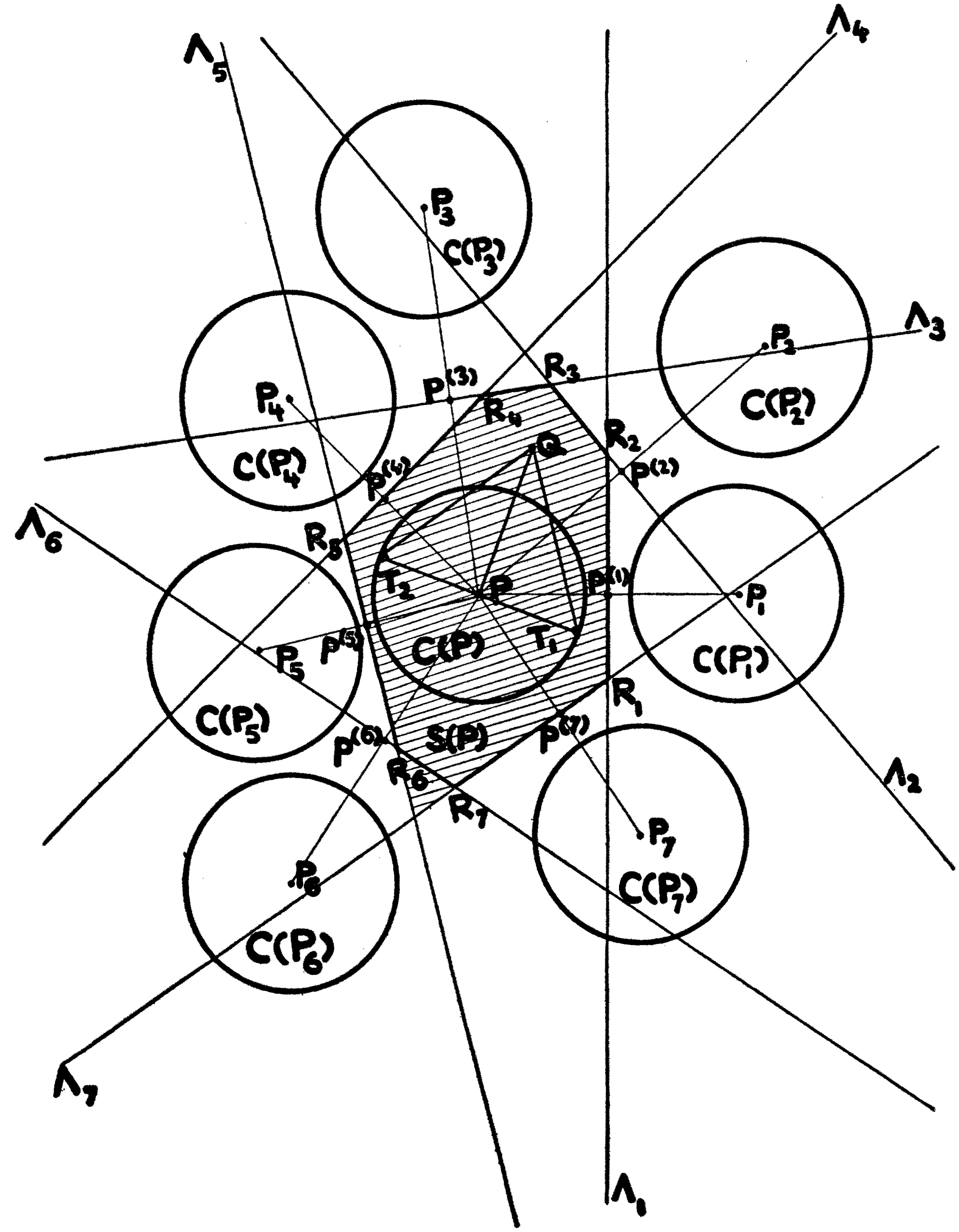


FIG. 1

We assume from now on that $S(P)$ is of *finite area*, $A(P)$ say. Let $Q \neq P$ be a point in $S(P)$, and denote by $T_1$ and $T_2$ the two endpoints of the diameter of $C(P)$ perpendicular to $PQ$. Since $S(P)$ is convex, and since the three points $Q$, $T_1$, $T_2$ belong to $S(P)$, the whole triangle $T_1QT_2$ is a subset of $S(P)$, and so its area is not greater than $A(P)$. Now the area of this triangle is

$$\tfrac{1}{2}\overline{T_1T_2} \times \overline{PQ} = \overline{PQ}, \quad \text{since} \quad \overline{T_1T_2} = 2,$$

whence

(2) $$\overline{PQ} \leq A(P) \quad \textit{for every point } Q \textit{ of } S(P).$$

By (2), $S(P)$ lies in the circle $C'(P)$ of center $P$ and radius $A(P)$. Hence it has points in common only with those lines $\Lambda_k$ which touch or pass through $C'(P)$. As $P^{(k)}$ is the point of $\Lambda_k$ nearest to $P$, this requires that

$$\overline{PP^{(k)}} \leq A(P), \quad \textit{i.e.}, \quad \text{that} \quad \overline{PP_k} = 2\overline{PP^{(k)}} \leqq 2A(P).$$

But since only a finite number of points $P_k$ of $\Sigma$ satisfy this inequality, the boundary of $S(P)$ meets only a finite number of the lines $\Lambda_k$. Therefore $S(P)$ *is a convex polygon*, say with the $n$ sides $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$, and the $n$ vertices $R_1, R_2$ $\cdots, R_n$. We choose the notation such that these sides and vertices lie on the boundary of $S(P)$ in the order of their indices, and that $R_k$ and $R_{k+1}$ are the vertices on $\Lambda_k$, and so $\Lambda_{k-1}$ and $\Lambda_k$ are the sides through $R_k$. (The indices 0 and $n+1$ must be replaced by $n$ and 1, respectively.)

**3. A fundamental lemma.** The lines from $P$ to the $n$ vertices $R_1, R_2, \cdots, R_n$ split $S(P)$ into the $n$ triangles

$$R_1PR_2, \; R_2PR_3, \cdots, \; R_nPR_1,$$

say of areas

$$a_1, \; a_2, \cdots, \; a_n.$$

Let further the angles at $P$ of these triangles be

$$\alpha_1, \; \alpha_2, \cdots, \; \alpha_n,$$

respectively. Then, evidently,

(3) $$a_1 + a_2 + \cdots + a_n = A(P),$$

and

(4) $$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 2\pi.$$

In the next paragraphs, we prove the following

LEMMA 1: *For every index $k$,*

(5) $$a_k \geqq \frac{\sqrt{3}}{\pi}\alpha_k,$$

*with equality if and only if the two circles $C(P)$ and $C(P_k)$ touch each other, and are both touched by the circles* $C(P_{k-1})$ and $C(P_{k+1})$.

For the proof of (5), put (Fig. 2)

$$x = \overline{PP^{(k)}} = \tfrac{1}{2}\overline{PP_k}, \quad \text{so that} \quad x \geqq 1.$$

Further denote by $\Gamma$ the circle of center $P$ and radius $x$; then $\Lambda_k$ is a tangent of

$\Gamma$. The two lines from $P$ to $R_k$ and $R_{k+1}$ cut off $\Gamma$ a sector of angle $\alpha_k$, hence of area $\frac{1}{2}x^2\alpha_k$. Since this sector lies entirely in the triangle $R_kPR_{k+1}$ of area $a_k$, the

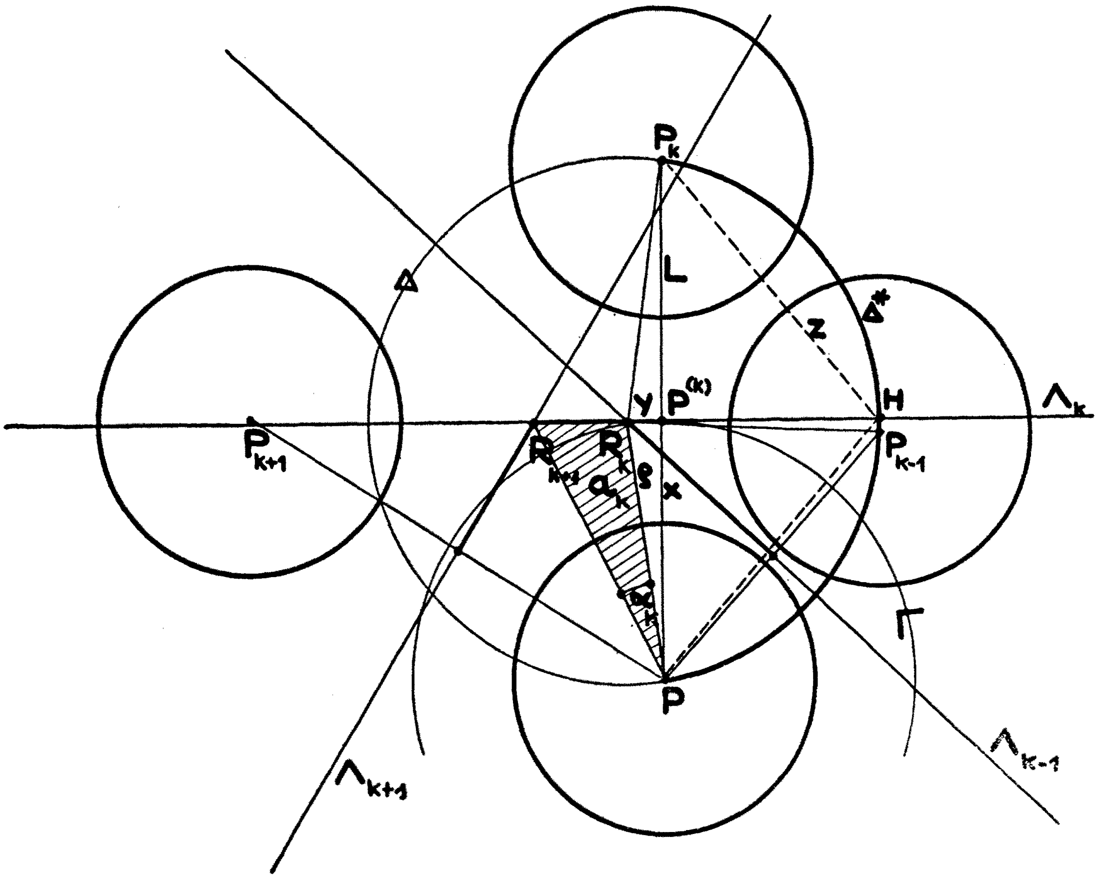


FIG. 2

inequality $a_k > \frac{1}{2}x^2\alpha_k$ holds. Hence if $\frac{1}{2}x^2 \geqq \sqrt{3}/\pi$, *i.e.*, if $x \geqq \sqrt[4]{12/\pi^2} > 1$, then (5) is satisfied with the sign "$>$" instead of "$\geqq$," and so the assertion is proved, since $C(P)$ and $C(P_k)$ obviously do not touch each other.

We may therefore exclude this case, and assume from now on that

$$1 \leqq x < \sqrt[4]{\frac{12}{\pi^2}} < \sqrt{\frac{4}{3}}. \tag{6}$$

We now prove the result:

*The point $P^{(k)}$ is interior to the segment $R_kR_{k+1}$.*

If this is not so, then $R_k$, $R_{k+1}$ are two distinct points of the line $\Lambda_k$, lying on the same side of $P^{(k)}$. Hence one of these points, say $R_k$, is nearer than the other one to $P^{(k)}$; and $R_k$ possibly coincides with $P^{(k)}$.

By our notation, $\Lambda_{k-1}$ and $\Lambda_k$ meet at $R_k$. Since all points of $\Lambda_{k-1}$ are equidistant from $P$ and $P_{k-1}$, this implies that $\overline{PR_k} = \overline{P_{k-1}R_k}$, and so $P$ and $P_{k-1}$ lie on the circle $\Delta$ of center $R_k$ and radius $\rho = \overline{PR_k}$. This circle contains also $P_k$, and is divided into two arcs by the line $L$ joining $P$ and $P_k$. Let $\Delta^*$ be the arc which meets $\Lambda_k$ at a point, $H$ say, separated from $R_{k+1}$ by $R_k$; then $\Delta^*$ is separated from $R_{k+1}$ by $L$. Since the only common point of the line $\Lambda_{k-1}$ and the angle $PR_kR_{k+1}$ is the vertex $R_k$ of this angle, it follows that the image $P_{k-1}$ of $P$ in $\Lambda_{k-1}$ is separated from $R_{k+1}$ by both the line $PR_k$ and its image $R_kP_k$ in $\Lambda_k = R_kR_{k+1}$; hence $P_{k-1}$ lies on $\Delta^*$. On putting

$$z = \overline{PH} = \overline{P_kH},$$

it is evident that for every point $Q$ on $\Delta^*$,

$$\min(\overline{PQ}, \overline{P_kQ}) \leqq z,$$

and so, in particular,

$$\min(\overline{PP_{k-1}}, \overline{P_kP_{k-1}}) \leqq z.$$

Therefore, by the definition of $\Sigma$,

$$z \geqq 2. \tag{7}$$

Next put

$$\overline{R_kP^{(k)}} = y,$$

so that

$$\rho^2 = x^2 + y^2 \quad \text{and} \quad \overline{P^{(k)}H} = \overline{R_kH} - \overline{R_kP^{(k)}} = \rho - y = \sqrt{x^2 + y^2} - y \leqq x,$$

since $\sqrt{x^2+y^2} \leqq x+y$. Hence

$$z^2 = \overline{PP^{(k)}}^2 + \overline{P^{(k)}H}^2 \leqq x^2 + x^2 = 2x^2,$$

whence, from (6),

$$z \leqq \sqrt{2}\, x < \sqrt{2}\sqrt{\tfrac{4}{3}} < 2,$$

contrary to (7). This contradiction proves the result.

From the above, the line $PP^{(k)}$ divides the triangle $R_kPR_{k+1}$ into the two triangles $R_kPP^{(k)}$, say of area $b_k$, and $P^{(k)}PR_{k+1}$, say of area $c_k$, so that

$$b_k + c_k = a_k. \tag{8}$$

The line $PP^{(k)}$ splits $\alpha_k$ into two angles $\beta_k = R_kPP^{(k)}$ and $\gamma_k = P^{(k)}PR_{k+1}$ satisfying

$$\beta_k + \gamma_k = \alpha_k. \tag{9}$$

By (8) and (9), Lemma 1 is proved if we can show that

$$b_k \geqq \frac{\sqrt{3}}{\pi}\beta_k, \tag{10}$$

and

$$c_k \geqq \frac{\sqrt{3}}{\pi}\gamma_k, \tag{11}$$

with equality if and only if $C(P)$ and $C(P_k)$ touch each other, and are touched by $C(P_{k-1})$ in the case of the inequality (10), and by $C(P_{k+1})$ in inequality (11).

It suffices to prove the assertion (10), since (11) can be treated likewise.

As before, we denote by $\Delta$ the circle of center $R_k$ and radius $\rho=\overline{PR_k}$ (Fig. 3); then $P$, $P_{k-1}$ and $P_k$ lie on this circle. The line $L$ joining $P$ and $P_k$ divides $\Delta$ into two unequal arcs (since $P^{(k)}$ and $R_k$ are distinct); let $\Delta^*$ be the larger one, *i.e.*

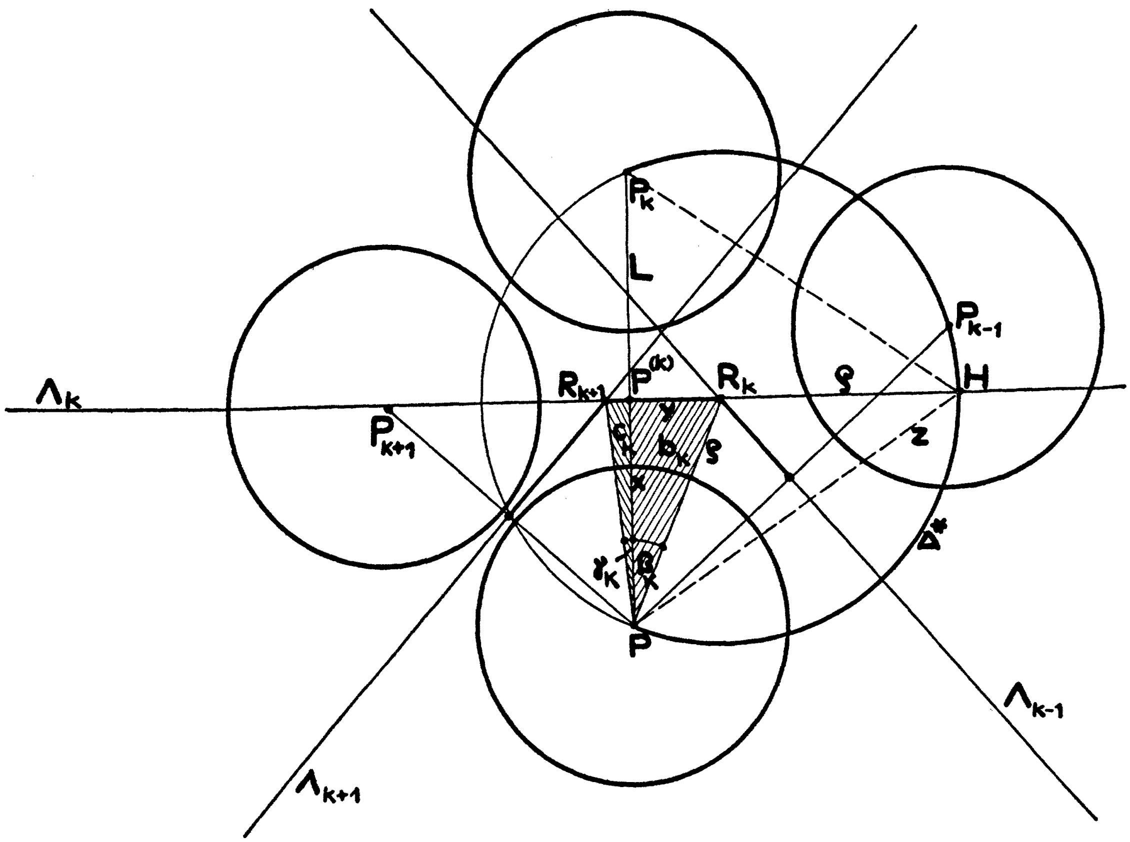


FIG. 3

that arc which is separated from $R_{k+1}$ by $L$. Then, by an argument already used we see that $P_{k-1}$ lies on $\Delta^*$.

Denote again by $H$ the point where $\Delta^*$ and $\Lambda_k$ intersect, and put

$$y = \overline{R_kP^{(k)}}, \qquad z = \overline{PH} = \overline{P_kH}.$$

Then $\rho=\sqrt{x^2+y^2}$, and we find as in §7 that

$$z \geqq 2, \tag{12}$$

but now

$$\begin{aligned} \overline{P^{(k)}H} &= \overline{R_kH} + \overline{P^{(k)}R_k} = \rho + y = \sqrt{x^2+y^2} + y, \\ z^2 &= \overline{PH}^2 = \overline{PP^{(k)}}^2 + \overline{P^{(k)}H}^2 = x^2 + (\sqrt{x^2+y^2}+y)^2 \\ &= 2(x^2+y^2+y\sqrt{x^2+y^2}). \end{aligned} \tag{13}$$

From (12) and (13),

$$x^2+y^2+y\sqrt{x^2+y^2} \geqq 2,$$

whence, on solving for $y$,

$$y \geqq \frac{2 - x^2}{\sqrt{4 - x^2}}.$$

Since further

$$b_k = \tfrac{1}{2}xy, \qquad \beta_k = \sin^{-1} \frac{y}{\sqrt{x^2 + y^2}},$$

the inequality (10) can be written as

$$F(x, y) \geqq 0, \quad \text{where} \quad F(x, y) = \pi xy - \sqrt{12} \sin^{-1} \frac{y}{\sqrt{x^2 + y^2}}.$$

Hence the assertion (10) is proved if we can show that

$$F(x, y) > 0, \tag{14}$$

if

$$1 \leqq x < \sqrt{\tfrac{4}{3}} \quad \text{and} \quad y \geqq \frac{2 - x^2}{\sqrt{4 - x^2}}, \quad \text{and either} \quad x > 1 \quad \text{or} \tag{15}$$
$$y > \frac{2 - x^2}{\sqrt{4 - x^2}}, \quad \text{or both.}$$

For in the excluded case, we have

$$x = 1, \qquad y = \frac{2 - x^2}{\sqrt{4 - x^2}} = \sqrt{\tfrac{1}{3}}, \qquad z = 2, \qquad F(x, y) = 0,$$

and so the equality sign holds in (10). Moreover, since $z = 2$, every point $Q \neq H$ on $\Delta^*$ has a distance less than 2 from either $P$ or $P_k$; therefore $P_{k-1}$ must coincide with $H$, *i.e.*, the three circles $C(P)$, $C(P_{k-1})$, $C(P_k)$ touch in pairs.

When (15) is satisfied, then

$$\frac{x^2 + y^2}{x} \frac{\partial F(x, y)}{\partial y} = \pi(x^2 + y^2) - \sqrt{12} \geqq \pi\left(x^2 + \frac{(2 - x^2)^2}{4 - x^2}\right) - \sqrt{12}$$
$$= \frac{4\pi}{4 - x^2} - \sqrt{12} \geqq \frac{4\pi}{4 - 1} - \sqrt{12} > 0.$$

Hence $F(x, y)$ is a *strictly increasing* function of $y$. To prove (14), it therefore suffices to show that the function

$$f(x) = F\left(x, \frac{2 - x^2}{\sqrt{4 - x^2}}\right) = \pi \frac{x(2 - x^2)}{\sqrt{4 - x^2}} - \sqrt{12} \sin^{-1} (1 - \tfrac{1}{2}x^2)$$

is *strictly increasing*, since $f(1) = 0$. Now

$$f(x) = 2g(t), \quad \text{where} \quad t = 1 - \tfrac{1}{2}x^2, \qquad g(t) = \pi t \sqrt{\frac{1 - t}{1 + t}} - \sqrt{3} \sin^{-1} t,$$

and where, by (15), $t$ satisfies the inequality

$$\tfrac{1}{3} = 1 - \tfrac{2}{3} < t < 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

On differentiating, we get

$$\frac{df(x)}{dx} = -2x\frac{dg(t)}{dt}, \qquad \frac{dg(t)}{dt} = \frac{1-t}{(1-t^2)^{3/2}}\left[\pi - (\pi t + \sqrt{3})(1+t)\right].$$

Now $3 < \pi < 10/3$ and $\sqrt{3} > 3/2$, hence

$$(\pi t + \sqrt{3})(1+t) \geqq (3\cdot\tfrac{1}{3} + \tfrac{3}{2})(1 + \tfrac{1}{3}) = \tfrac{10}{3} > \pi,$$

and therefore

$$\frac{dg(t)}{dt} < 0, \qquad \frac{df(x)}{dx} > 0,$$

as asserted. This concludes the proof of Lemma 1.

**4. A second lemma.** We now prove the following

LEMMA 2. *The convex polygon $S(P)$, defined in* §2, *is of area*

$$A(P) \geqq \sqrt{12}, \tag{16}$$

*with equality if and only if $C(P)$ is touched by six circles $C(P_1), \cdots, C(P_6)$, whose centers form a regular hexagon of side* 2 *and center $P$.*

From (3), (4), and Lemma 1, the inequality (16) follows at once. We see moreover, that the equality sign can hold only if

$$a_k = \frac{\sqrt{3}}{\pi}\alpha_k \qquad (k = 1, 2, \cdots, n),$$

*i.e.*, if each circle $C(P_k)$ in the set

$$C(P_1), C(P_2), \cdots, C(P_n)$$

is touched by both $C(P_{k-1})$ and $C(P_{k+1})$, and itself touches $C(P)$. Hence

$$\overline{PP_1} = \overline{PP_2} = \cdots = \overline{PP_n} = \overline{P_1P_2} = \overline{P_2P_3} = \cdots = \overline{P_nP_1} = 2.$$

Now the regular hexagon, and no other regular polygon, has the property that its side is equal to the radius of the circumscribed circle; therefore $n$ must be equal to 6, and the circles $C(P), C(P_1), \cdots, C(P_6)$ must be situated as asserted.

**5. The theorem.** We can now prove the following theorem.

*Let $\Pi_0$ be a convex polygon of angles not greater than $\frac{2}{3}\pi$, hence of at most six sides. If $A$ denotes the area of $\Pi_0$, then at most $A/\sqrt{12}$ circles of radius* 1 *can be placed in $\Pi_0$ such that no two of these circles overlap.*

*Proof* (Fig. 4): Let $C_{01}, C_{02}, \cdots, C_{0l}$ be the circles placed in $\Pi_0$, and let $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ be the sides of $\Pi_0$. Denote further by $\Pi_h$ the polygon symmetrical to $\Pi_0$ in $\Lambda_h$, and by $C_{h1}, C_{h2}, \cdots, C_{hl}$ the circles symmetrical to $C_{01}, C_{02}, \cdots, C_{0l}$ in $\Lambda_h$; these new circles lie in $\Pi_h$. No two of the $m=(n+1)l$ circles

$$C_{01}, C_{02}, \cdots, C_{0l}, \cdots, C_{n1}, C_{n2}, \cdots, C_{nl} \tag{17}$$

overlap. This is obvious, from the hypothesis, for any two circles in the *same* polygon $\Pi_h$, and also for any two circles one of which lies in $\Pi_0$. To prove the assertion for two circles lying in two different polygons $\Pi_1, \Pi_2, \cdots, \Pi_n$, it is obviously sufficient to show that no two of these polygons, $\Pi_h$ and $\Pi_k$ say $(h \neq k)$, overlap. This is evident if $\Lambda_h$, $\Lambda_k$ are parallel, since then $\Pi_h$, $\Pi_k$ lie on opposite

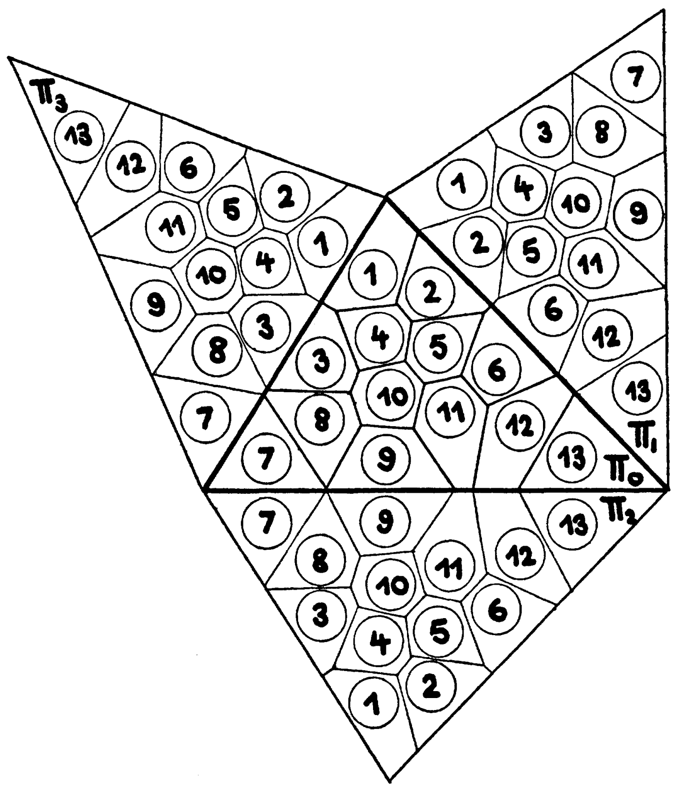


FIG. 4

sides of the strip determined by these lines. If $\Lambda_h$, $\Lambda_k$ meet at a point, $O$ say, we consider that angle $\phi$ of vertex $O$ formed by these two lines which contains $\Pi_0$. Then $\phi \leqq \frac{2}{3}\pi$, from the hypothesis for $\Pi_0$, and so the images of $\phi$ in $\Lambda_h$ and $\Lambda_k$ are two non-overlapping angles of vertex $O$. Since one of these angles contains $\Pi_h$ and the other contains $\Pi_k$, it follows that $\Pi_h$ and $\Pi_k$ do not overlap.

Denote now by $\Sigma$ the set of the centers

$$P_{01}, P_{02}, \cdots, P_{0l}, \cdots, P_{n1}, P_{n2}, \cdots, P_{nl}$$

of the circles (17), so that $\Sigma$ has the properties stated in §1. In accordance with the definition in §1, we form the convex regions

$$S(P_{01}), \cdots, S(P_{0l}), \cdots, S(P_{n1}), \cdots, S(P_{nl});$$

these $m$ regions together fill out the plane, and no two of them overlap.

We show next that all $l$ polygons

$$S(P_{01}), S(P_{02}), \cdots, S(P_{0l}) \tag{18}$$

lie in $\Pi_0$. For consider, *e.g.* $S(P_{01})$, and take any side $\Lambda_h$ of $\Pi_0$. If $P_{h1}$ is the point symmetrical to $P_{01}$ in $\Lambda_h$, then every point $Q$ of $S(P_{01})$ is at least as near to $P_{01}$ as to $P_{h1}$, hence lies on the same side of $\Lambda_h$ as $P_{01}$. This is true for all indices $h=1, 2, \cdots, n$, and so $Q$ lies in $\Pi_0$. We deduce then that the total area of the $l$ polygons (18) is just equal to the area $A$ of $\Pi_0$.

By Lemma 2, each polygon $S(P_{0i})$ is at least of area $\sqrt{12}$. This implies that

$$l\sqrt{12} \leqq A, \qquad l \leqq \frac{A}{\sqrt{12}},$$

as asserted. Moreover, our proof shows that *the equality*

$$l = \frac{A}{\sqrt{12}}$$

*holds, if and only if* $\Pi_0$ *is the sum set of a finite number of regular hexagons of side* 2.

We finally remark that the Theorem is untrue for polygons with angles greater than $\frac{2}{3}\pi$. For instance, the regular heptagon circumscribed to a circle of radius 1 is of area less than $\sqrt{12}$.

## AN APPLICATION OF A THEOREM OF SYLVESTER

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In this paper a little used theorem on determinants due to Sylvester is employed to prove a theorem in the geometry of curves which generalizes the well known result that the tangent plane to a developable surface is the same at all points of a fixed generator.*

**1. The theorem of Sylvester.** Let a matrix $(a_{ij}; i,j=0, \cdots, m)$ be given, and define

$$A_h \equiv \begin{vmatrix} a_{00} & \cdots & a_{0h} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ a_{h0} & \cdots & a_{hh} \end{vmatrix} \qquad (h = 0, \cdots, m), A \equiv A_m,$$

$$b_{rs}^{(h)} \equiv \begin{vmatrix} a_{00} & \cdots & a_{0h} & a_{0s} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ a_{h0} & \cdots & a_{hh} & a_{hs} \\ a_{r0} & \cdots & a_{rh} & a_{rs} \end{vmatrix} \qquad (r, s = h+1, \cdots, m),$$

$$B^{(h)} \equiv \left| b_{rs}^{(h)} \right| \qquad (h = 0, \cdots, m-1).$$

* E. P. Lane, Projective Differential Geometry of Curves and Surfaces, Chicago, 1932, pp. 37–38.