by (4·1). Finally
$$0 \leqq \Delta^2\lambda_m \leqq \frac{C}{(m+1)^2 (lm)^{\frac{3}{2}}},$$

and so
$$|P_N| \leqq C \operatorname*{Max}_{m \leqq N} |\sigma_m(\theta)| \sum_0^\infty \frac{1}{(m+1)(lm)^{\frac{3}{2}}} \leqq C \operatorname{Max} |\sigma_n(\theta)|,$$

$$\int P_N^2 \, d\theta \leqq C \int \operatorname{Max} \sigma_n^2(\theta) \, d\theta \leqq C, \tag{4·5}$$

again by (4·1). Finally, (9) follows from (4·2)–(4·5).

REFERENCES

(1) HARDY, G. H. and LITTLEWOOD, J. E. *Acta Math.* 54 (1930), 81–116.
(2) KOLMOGOROFF, A. and SELIVERSTOFF, G. *C.R. Acad. Sci., Paris*, 178 (1925), 303–5.
(3) KOLMOGOROFF, A. and SELIVERSTOFF, G. *Rend. Accad. Lincei*, 3 (1926), 307–10.
(4) LITTLEWOOD, J. E. and PALEY, R. E. A. C. *Proc. London Math. Soc.* (2), 43 (1937), 105–26.
(5) PLESSNER, A. *J. reine angew. Math.* 155 (1925), 15–25.
(6) ZYGMUND, A. *Trigonometrical Series* (Warsaw, 1935).

TRINITY COLLEGE
CAMBRIDGE

ON LATTICE POINTS IN THE DOMAIN $|xy| \leqslant 1$, $|x+y| \leqslant \sqrt{5}$, AND APPLICATIONS TO ASYMPTOTIC FORMULAE IN LATTICE POINT THEORY (I)

BY K. MAHLER

Communicated by L. J. MORDELL

Received 25 June 1943

Preface. In order to make this paper intelligible to the reader, I repeat the following definitions and results from my paper 'On lattice points in star domains', which is to appear in the *Proceedings of the London Mathematical Society.*

A *finite* star domain K in the (x, y)-plane is a bounded closed point set such that

(1) the origin $O = (0, 0)$ is an *inner* point of K;

(2) K is symmetrical in O;

(3) the boundary L of K is a Jordan curve;

(4) every radius vector from O intersects L in just one point.

The set K is called an *infinite* star domain, if for every $r > 0$ the subset of its points (x, y) with $x^2 + y^2 \leqslant r^2$ forms a finite star domain.

The lattice

$$(\Lambda) \qquad x = \alpha h + \beta k, \quad y = \gamma h + \delta k \quad (h, k = 0, \pm 1, \pm 2, \ldots),$$

of determinant
$$d(\Lambda) = |\alpha\delta - \beta\gamma|,$$

is called *K-admissible*, if O is the only inner point of K contained in Λ. The lower bound $\Delta(K)$ of $d(\Lambda)$ extended over all K-admissible lattices is positive. There exist *critical lattices* of K, i.e. K-admissible lattices such that $d(\Lambda) = \Delta(K)$.

For finite star domains, the following results hold.

A critical lattice has at least *four* points on L; if it has only four such points, then it is called *singular*. If $\pm P_1$ and $\pm P_2$ are the four points of a singular lattice on L, then the sides of the parallelogram with vertices at $\pm P_1 \pm P_2$ are *tac-lines* of L at $\pm P_1$ and $\pm P_2$ (a tac-line at a point P of L is a line through P such that all points of K sufficiently near to P lie on one side of the line).

If P_1, P_2 are two independent points of a K-admissible lattice Λ such that the line segment joining them consists only of *inner* points of K, then P_1, P_2 form a basis of Λ.

As usual, when $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are any two points, and u and v any two real numbers, then $uP_1 + vP_2$ denotes the point

$$uP_1 + vP_2 = (ux_1 + vx_2,\ uy_1 + vy_2),$$

and (P_1, P_2) is the determinant $\qquad (P_1, P_2) = x_1 y_2 - x_2 y_1.$

A well-known theorem of Hurwitz states that if K' is the *infinite* star domain

$$|xy| \leqslant 1,$$

then $$\Delta(K') = \sqrt{5}.$$

I prove in the first part of this paper that the *finite* star domain

$$(K) \qquad |xy| \leqslant 1, \quad |x+y| \leqslant \sqrt{5},$$

which is contained in K', has the same minimum determinant†

$$(A) \qquad \Delta(K) = \sqrt{5}.$$

Moreover, I also show that K is *a minimum* subset of K' with $\Delta(K) = \sqrt{5}$, i.e. that, if $H \neq K$ is any star domain contained in K, then

$$\Delta(H) < \Delta(K).$$

In the second part, I apply (A) to find an asymptotic formula for the minimum determinant $\Delta(G)$ of the domain

$$(G) \qquad |x|^{\alpha} + |y|^{\alpha} \leqslant 1,$$

where α is a small positive number which tends to zero. This domain was first studied by Mordell when $\alpha \leqslant 1$; he succeeded in obtaining the exact value of $\Delta(G)$ for all α with $1 \geqslant \alpha \geqslant 0{\cdot}33$ approximately. But for smaller values of α, the critical lattices take different forms according to the intervals in which α lies, and then the problem becomes far more difficult. It is interesting that $\Delta(G)$ has a simple asymptotic formula, namely

$$\Delta(G) \sim 2^{-2/\alpha}\sqrt{5}.$$

I further apply (A) to find an asymptotic formula for the minimum of a positive definite binary quartic form $f(x, y)$ for integral x, y not both zero, when the discriminant of $f(x, y)$ is given and the absolute invariant tends to infinity.

Another application of (A) is made in a separate note 'On lattice points in infinite star domains', which is to appear in the *Journal of the London Mathematical Society*. I show that critical lattices of an *infinite* star domain need not have any points on the boundary of this domain.

Let K be the domain $\qquad |xy| \leqslant 1, \quad |x+y| \leqslant \sqrt{5}.$

The boundary L of K consists of

(1) the arc L_1 of $xy = 1$ between the two points

$$P_1 = \left(\frac{1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right), \quad P_2 = \left(\frac{-1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right),$$

both points included;

† Hurwitz's theorem $\Delta(K') = \sqrt{5}$ follows easily from (A). Moreover (A) leads to a simple algorithm for finding any number of points of Λ in K', if $d(\Lambda) \leqslant \sqrt{5}$.

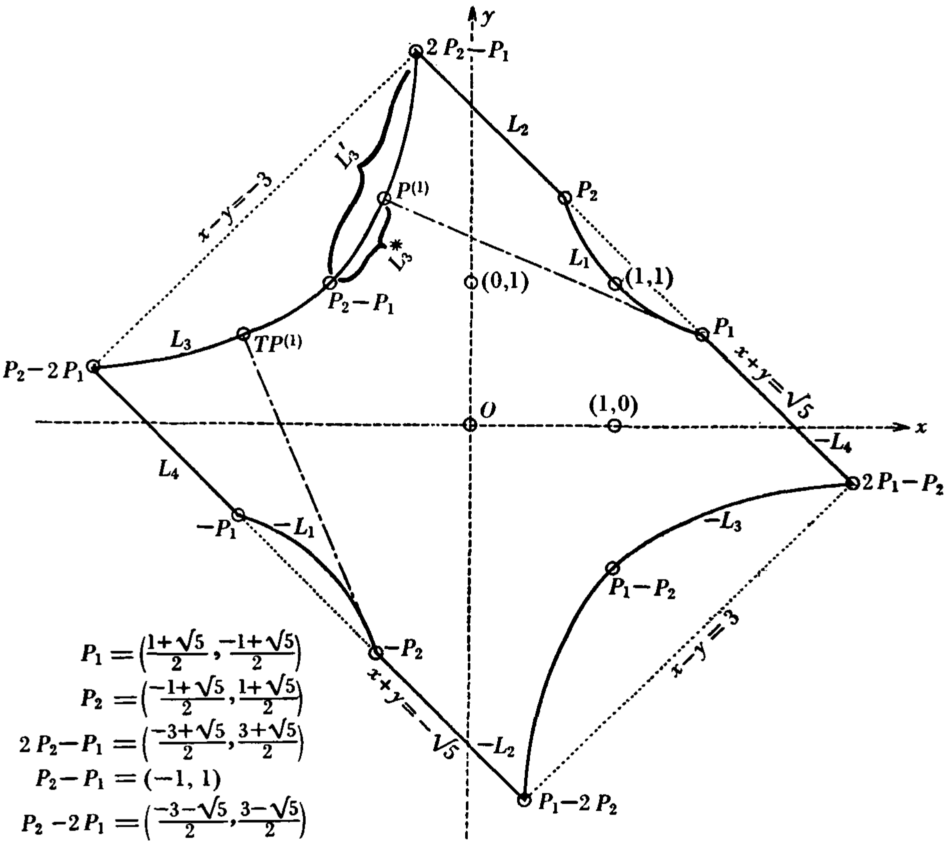
(2) the line segment L_2 connecting P_2 and $2P_2-P_1$, both points excluded;

(3) the arc L_3 of $xy=-1$ between $2P_2-P_1$ and P_2-2P_1, both points included;

(4) the line segment L_4 connecting P_2-2P_1 and $-P_1$, both points excluded;

(5) the arcs and line segments $-L_1$, $-L_2$, $-L_3$, $-L_4$ symmetrical to L_1, L_2, L_3, L_4 in the origin $O=(0,0)$.

Our aim is to prove the following theorems.



THEOREM 1. *$\Delta(K)=\sqrt5$; i.e. every lattice*

$$(\Lambda)\qquad x=\alpha h+\beta k,\quad y=\gamma h+\delta k\quad (h,k=0,\pm1,\pm2,\dots)$$

of determinant $$d(\Lambda)=|\alpha\delta-\beta\gamma|\leqslant\sqrt5$$

contains at least one point of K different from O; but lattices of greater determinant need not have this property.

THEOREM 2. *The domain K has the following critical lattices and no further ones.*

(1) *The lattice Λ_0 generated by the two points P_1 and P_2; it has ten points on L, namely $\pm P_1$, $\pm P_2$, $\pm(2P_2-P_1)$, $\pm(P_2-P_1)$, $\pm(P_2-2P_1)$.*

(2) *The lattices Λ_1 generated by an arbitrary point Q_2 on L_2 and by the point P_2-P_1; it has six points on L, namely $\pm Q_2$, $\pm(P_2-P_1)$, $\pm(2P_2-2P_1-Q_2)$.*

(3) *The lattices Λ_2 generated by an arbitrary point Q_1 on L_1 and by that point Q_3' on L_3 for which also Q_1+Q_3' is a point on L_3; it has six points on L, namely $\pm Q_1$, $\pm Q_3'$, $\pm(Q_1+Q_3')$.*

THEOREM 3. *If the star domain H is contained in K, but is not identical with K, then $\Delta(H)<\Delta(K)=\sqrt{5}$.*

Proof of Theorem 1 *and Theorem* 2

Denote from now on by Λ an arbitrary critical lattice of K. The existence of the K-admissible lattices Λ_0, Λ_1, Λ_2 of determinant $\sqrt{5}$ shows that necessarily $d(\Lambda)\leqslant\sqrt{5}$.

We distinguish the following three cases:

(*a*) *Λ contains points of both L_2 and L_4;*

(*b*) *Λ contains points of at most one of the two line segments L_2 and L_4;*

(*c*) *no point of Λ lies on either L_2 or L_4.*

There is no need to mention here possible lattice points on L_1 and L_3.

Before discussing these different cases, we make the obvious remark that each line segment L_2 and L_4 can contain *at most one* point of Λ. For if there were two points on one of them, then the difference point would clearly be an *inner* point of the line segment connecting P_2-P_1 with P_1-P_2, and so it would be an inner point of K, contrary to hypothesis.

Case a

Denote by $Q_2=(x_2,y_2)$ the point of Λ on L_2, by $Q_4=(x_4,y_4)$ the point on L_4. The line L_2 clearly is not parallel to the line through O and Q_4, nor is the line L_4 parallel to the line through O and Q_2. Hence Λ cannot be a *singular* lattice, and so has *at least one* further point on either L_1 or L_3; it may even have points on both arcs. We distinguish now two cases, according as Λ has, or has not, a point on L_3.

Subcase a, 1. *Λ has a point $Q_3=(x_3,y_3)$ on L_3, and possibly further points on L_1 or L_3.*

Evidently K and L are transformed into themselves by the reflexion

$$(T)\qquad x\to -y,\quad y\to -x$$

in the line $x+y=0$. We may therefore assume without loss of generality that Q_3 lies on that arc L_3' of L_3 for which $x+y\geqslant 0$. Now the tangent to $xy=1$ at P_1 intersects $xy=-1$ at the point

$$P^{(1)}=\{\tfrac{1}{2}[(1+\sqrt{5})(1-\sqrt{2})],\ \tfrac{1}{2}[(-1+\sqrt{5})(1+\sqrt{2})]\}=(x^{(1)},y^{(1)}),$$

say. For the tangent at P_1 is

$$\frac{x}{x_1}+\frac{y}{y_1}=2,$$

and clearly contains $P^{(1)}$, which also lies on $xy=-1$.

Similarly, the tangent to $xy=1$ at $-P_2$ intersects $xy=-1$ at

$$TP^{(1)}=(-y^{(1)},-x^{(1)}),$$

as follows on applying the reflexion T in $x+y=0$.

Denote by L_3^* that closed arc of L_3' which is bounded by the two points P_2-P_1 and $P^{(1)}$. *Then Q_3 must lie on L_3^*.* For otherwise Q_3 lies on the part of L_3' between $P^{(1)}$ and $2P_2-P_1$, and so evidently

$$x_3+y_3\geqslant x^{(1)}+y^{(1)}=\sqrt{5}-\sqrt{2}.$$

On the other hand, since Q_2 lies on L_2, $x_2+y_2 = \sqrt{5}$. Hence the coordinates of Q_2-Q_3, (ξ, η) say, satisfy the inequality

$$\xi+\eta \leqslant \sqrt{5}-(\sqrt{5}-\sqrt{2}) = \sqrt{2}.$$

But this inequality leads to a contradiction. For both Q_2 and Q_3 are points of the triangle with vertices at P_2, $2P_2-P_1$, P_2-P_1; and Q_2, but not Q_3, lies on the basis side $x+y = \sqrt{5}$ of this isosceles triangle. Hence, by the translation which adds P_1-P_2 to every point of the plane, Q_2-Q_3 is a point different from $\pm(P_1-P_2)$ of the *larger* quadrilateral cut off by the line $x+y = \sqrt{2}$ from the quadrilateral with vertices at P_1-P_2, P_1, P_2, P_2-P_1. It therefore is an inner point of K, since, for all points of L,

$$x+y \geqslant 2 > \sqrt{2}.$$

Since Q_3 lies on L_3^*, both Q_2, Q_3 and also Q_3, Q_4 form a basis of Λ. For the line segments joining any point on L_3^* with any point on $-L_2$ or on $-L_4$ consist only of *inner* points of K, except for their end points; the assertion is therefore immediate (see the preface).

Hence there are positive integers u_2, v_2, u_4, v_4 such that

$$Q_2 = u_2 Q_3 - v_2 Q_4, \quad Q_4 = -u_4 Q_2 + v_4 Q_3,$$

and so obviously $u_4 = v_2 = 1$; and therefore

$$Q_3 = (Q_2+Q_4)/g,$$

where $g = u_2 = v_4$ is a positive integer. *This integer cannot be greater than* 2. For both Q_2 and Q_4 lie below the line $y-x = 3$ through $2P_2-P_1$ and P_2-2P_1; hence Q_3, a point on the radius vector from O through P_2-P_1, lies below the line

$$y-x = (3+3)/g.$$

If now $g \geqslant 3$, then this means that $y_3-x_3 < 2$; hence Q_3 is an inner point of the line segment from O to P_2-P_1, i.e. an inner point of K, contrary to hypothesis.

If, next, $g = 1$, then $Q_3 = Q_2+Q_4$, and so also Q_2 and Q_4 form a basis of Λ. But it is clear from the figure that

$$d(\Lambda) = (Q_2, Q_4) > (P_2, -P_1) = \sqrt{5},$$

and so Λ is not critical.

If, lastly, $g = 2$, then $Q_3 = \frac{1}{2}(Q_2+Q_4)$ lies on the radius vector from O through P_2-P_1; it must therefore be the point P_2-P_1. Then (Q_2, Q_4) becomes independent of the positions of Q_2 and Q_4, in fact

$$(Q_2, Q_4) = (P_2, P_2-2P_1) = 2\sqrt{5}.$$

Hence always

$$d(\Lambda) = \tfrac{1}{2}(Q_2, Q_4) = \sqrt{5},$$

and Λ is a lattice of the type Λ_1.

Subcase a, 2. Λ has no points on L_3, but has a point $Q_1 = (x_1, y_1)$ on L_1, and possibly further points on L_1.

By an earlier remark, Λ possesses just one point $Q_2 = (x_2, y_2)$ on L_2 and just one point $Q_4 = (x_4, y_4)$ on L_4. If Λ has two points Q_1, Q_1' on L_1, then Q_1-Q_1' is clearly an inner point of K, except when these two points lie at P_1 and P_2, respectively, and then $\Lambda = \Lambda_0$.

Let us therefore assume from now on that Q_1 on L_1, Q_2 on L_2, Q_4 on L_4, and the symmetrical points $-Q_1$, $-Q_2$, $-Q_4$, are the only points of Λ on L. There is no restriction in assuming further that Q_1 lies on that arc L_1^* of L_1 which is bounded by the two points $(1, 1)$ and P_2; for otherwise it suffices to apply the reflexion

$$(-T) \qquad x \to y, \quad y \to x,$$

in order to change Λ into a new lattice $-T\Lambda$ of this type.

We then show that no such lattice can be critical. Let R be a lattice point which, together with Q_4, forms a basis of Λ. Then, for integral g and $h = \pm 1$, another basis is given by Q_4 and $S = gQ_4 + hR$. If now g and h are chosen suitably, then S is an inner point of the angle $(-Q_4)\,OQ_1$. For we may take the sign in $h = \pm 1$ so that hR lies above the line through O and Q_4, and then choose g so that $S = gQ_4 + hR$ lies to the right of the line through O and Q_1.

Hence $$Q_1 = u_1 Q_4 + v_1 S, \quad Q_2 = u_2 Q_4 + v_2 S,$$

where u_1, v_1, u_2, v_2 are *positive* integers. Let $\epsilon > 0$ be a sufficiently small number, and denote by Λ^* the lattice generated by the two points

$$Q_4^* = Q_4, \quad S^* = S - \epsilon Q_4.$$

The new lattice has the same determinant as Λ, since

$$d(\Lambda^*) = (S^*, Q_4^*) = (S, Q_4) = d(\Lambda).$$

It has, however, only the two points $\pm Q_4^*$ on L, and contains no inner points of K different from O, since to $\pm Q_1$, $\pm Q_2$ there correspond in Λ^* the new points

$$\pm Q_1^* = \pm(u_1 Q_4^* + v_1 S^*), \quad \pm Q_2^* = \pm(u_2 Q_4^* + v_2 S^*),$$

i.e. $$\pm Q_1^* = \pm(Q_1 - v_1 \epsilon Q_4), \quad \pm Q_2^* = \pm(Q_2 - v_2 \epsilon Q_4),$$

which clearly lie outside K, since the terms $-v_1\epsilon Q_4$, $-v_2\epsilon Q_4$ imply a translation to the right. But a lattice with only two points on L cannot be critical.

Case b

The affine transformation of determinant unity

$$(\Omega_t) \qquad x' = tx, \quad y' = \frac{1}{t}y \quad (t > 0)$$

leaves invariant any hyperbola $xy = \text{const.}$, and hence transforms

all points on $\pm L_2$ into points *outside* K if $t < 1$,
all points on $\pm L_4$ into points *outside* K if $t > 1$.

Hence if the critical lattice Λ has points on L_2, but not on L_4, then $\Omega_t \Lambda$ is a critical lattice with no points on either L_2 or L_4, if $t < 1$ and t is sufficiently near to 1. The same is true when Λ has points on L_4 but not on L_2, if we take $t > 1$ and sufficiently near to 1. Hence every critical lattice of the present case is derivable from one of the next case by a transformation Ω_t with $t > 0$.

Case c

Let Λ be a critical lattice with points on one or both of $\pm L_1$ and $\pm L_3$. If the lattice has points only on $\pm L_1$, then the affine transformation

$$x' + y' = t(x + y), \quad x' - y' = t^{-1}(x - y)$$

of determinant 1 changes Λ into a K-admissible lattice provided that $t > 1$ and that

$|t-1|$ is sufficiently small; this lattice will have no points on L, since distances parallel to $x = y$ are increased. If the lattice has points only on $\pm L_3$, then the same transformation changes Λ into a K-admissible lattice without points on L, if now $t<1$ and $|t-1|$ is sufficiently small. Hence a contradiction is obtained in both cases, and so Λ contains at least one point $Q_1 = (x_1, y_1)$ on L_1, and at least one point $Q_3 = (x_3, y_3)$ on L_3.

The lattice Λ cannot be singular. For suppose, first, that Q_1 is *not* an end-point of L_1. Then, by the general property of singular lattices, the tangent to $xy = 1$ at Q_1 must be parallel to OQ_3. Hence

$$-\frac{1}{x_1^2} = \frac{y_3}{x_3} = -\frac{1}{x_3^2},$$

and, since evidently x_1 is positive and x_3 negative, $x_3 = -x_1$, $y_3 = y_1$. Therefore, if $x_1 \geqslant 1$, then $Q_1 + Q_3 = (0, 2/x_1)$ is an inner point of K, and if $x_1 \leqslant 1$, then $Q_1 - Q_3 = (2x_1, 0)$ is an inner point of K; for $(0, 2)$ and $(2, 0)$ are inner points of K, and so also are all points of the line segments connecting them with O.

If, secondly, Q_1 lies at P_1 or P_2, and Q_3 is an *inner* point of L_3, then the tangent to $xy = -1$ at Q_3 must be parallel to OQ_1. Hence, as before, $x_3 = -x_1$, $y_3 = y_1$, and so $Q_1 + Q_3$ for $x_1 \geqslant 1$, or $Q_1 - Q_3$ for $x_1 \leqslant 1$, is an inner point of K.

Finally, it is not possible that Q_1 should be an end-point of L_1, and Q_3 an end-point of L_3. For then either Λ would be identical with Λ_0 and so possess ten points on L; or it would be a sublattice of Λ_0, and so be of determinant greater than $\sqrt{5}$.

We conclude then that, since Λ is not singular, it has at least *one* further point different from Q_1 and Q_3 on either L_1 or L_3.

If this further point lies on L_1, then the lattice has two points on L_1, and so, as in subcase a, 2, $\Lambda = \Lambda_0$.

We assume therefore from now on that this further point, say $Q_3' = (x_3', y_3')$, lies on L_3, so that Λ has *one* point Q_1 on L_1 and *two* points Q_3 and Q_3' on L_3. This does not exclude the possibility that L_3 contains further lattice points different from Q_3 and Q_3'.

Let the notation from now on be such that Q_1, Q_3, Q_3' follow on L in this order in the positive direction. We distinguish then two cases according as Q_3, Q_3' do, or do not, form a basis of Λ.

Subcase c, 1: *Q_3 and Q_3' form a basis of Λ.*

Evidently, for all points of L_3,

$$2 \leqslant y - x \leqslant 3,$$

with $y-x$ assuming its minimum at $P_2 - P_1$ and its maximum at $2P_2 - P_1$ and $P_2 - 2P_1$. Hence, in particular,

$$2 \leqslant y_3 - x_3 \leqslant 3, \quad -3 \leqslant -(y_3' - x_3') \leqslant -2.$$

Write $Q_3 - Q_3' = Q_0 = (x_0, y_0)$. Then by the last inequalities, $-1 \leqslant y_0 - x_0 \leqslant 1$. Further, the end-points P_1, P_2 of L_1 lie on the lines $y - x = \pm 1$, and so $-1 \leqslant y - x \leqslant 1$ for all points of L_1. Hence, since Q_0 cannot be an inner point of K, it lies either *on* or *outside* the hyperbola $xy = 1$, and so

$$x_0 y_0 = (x_3 - x_3')(y_3 - y_3') \geqslant 1.$$

Since Q_3 and Q_3' are points of the hyperbola $xy = -1$, this inequality can be written as

$$\frac{x_3}{x_3'} + \frac{x_3'}{x_3} \geqslant 3,$$

with the equality sign only if $Q_0 = Q_3 - Q_3'$ lies on L_1, that is, if $Q_3 - Q_3' = Q_1$, since Q_1 is the only lattice point on L_1.

Since Q_3, Q_3' form a basis of Λ,

$$d(\Lambda) = (Q_3, Q_3') = x_3 y_3' - x_3' y_3 = \frac{x_3'}{x_3} - \frac{x_3}{x_3'},$$

and so, by the identity
$$\left(\frac{x_3'}{x_3} - \frac{x_3}{x_3'}\right)^2 = \left(\frac{x_3}{x_3'} + \frac{x_3'}{x_3}\right)^2 - 4,$$

we finally get
$$d(\Lambda) \geqslant \sqrt{(3^2 - 4)} = \sqrt{5}.$$

Here the equality sign holds, as already mentioned, only if $Q_1 = Q_3 - Q_3'$, and then Λ is a lattice of the type Λ_2.

Subcase c, 2. Q_3 and Q_3' do not form a basis of Λ.

This implies (see the preface) that *the line joining $-Q_3'$ with Q_3 meets the arc L_1 in at least one real point.*

Assume that the positive integer n is defined by

$$(Q_3, Q_3') = nd(\Lambda).$$

By our hypothesis, this integer cannot be less than 2; we now show that it cannot be greater than 3.

The entire square
$$|x+y| \leqslant 2, \quad |x-y| \leqslant 2$$

of area 8 belongs to K; hence by Minkowski's theorem on linear forms,

$$\Delta(K) \geqslant 8/2^2 = 2,$$

and therefore
$$(Q_3, Q_3') = nd(\Lambda) \geqslant n\Delta(K) \geqslant 2n.$$

On the other hand, it is evident that (Q_3, Q_3') assumes its maximum when

$$Q_3 = 2P_2 - P_1, \quad Q_3' = P_2 - 2P_1,$$

and then
$$(Q_3, Q_3') = (2P_2 - P_1, P_2 - 2P_1) = 3\sqrt{5}.$$

Hence
$$2n \leqslant 3\sqrt{5}, \quad n \leqslant \tfrac{3}{2}\sqrt{5} < 4, \quad n \leqslant 3,$$

as asserted.

Let, first, $n = 3$. Then by Minkowski's method of adaptation of lattices, there exists a lattice point of the form
$$R = \tfrac{1}{3}(Q_3 - gQ_3') = (\xi, \eta),$$

say, in the parallelogram O, $-Q_3'$, $Q_3 - Q_3'$, Q_3; here $g = 0$, 1, or 2. The first possibility $g = 0$ can be excluded at once, since all inner points of the line segment OQ_3 are inner points of K.

Also the second possibility $g = 1$ leads to a contradiction. For

$$x_3 + y_3 \leqslant \sqrt{5}, \quad -(x_3' + y_3') \leqslant \sqrt{5}.$$

Hence
$$\xi + \eta \leqslant \tfrac{1}{3}(\sqrt{5} + \sqrt{5}) < 2,$$

and so R would again be an inner point of K.

Hence we must suppose that $g = 2$, so that both of

$$R = \tfrac{1}{3}(Q_3 - 2Q_3'), \quad S = \tfrac{1}{3}(2Q_3 - Q_3') = Q_3 - Q_3' - R$$

are lattice points. These two points R, S lie on the line segment joining $-Q_3'$ with Q_3 and divide this segment into three equal parts. It is clear that, for all points of this line segment, $x+y \leqslant \sqrt{5}$, since this inequality is satisfied by both $-Q_3'$ and Q_3. Hence R and S lie either *on* or *outside* the hyperbola $xy = 1$, and so

$$\frac{x_3-2x_3'}{3}\,\frac{y_3-2y_3'}{3} = \frac{x_3-2x_3'}{3}\,\frac{-x_3^{-1}+2x_3'^{-1}}{3} \geqslant 1$$

or
$$\frac{x_3}{x_3'}+\frac{x_3'}{x_3} \geqslant \frac{9+4+1}{2} = 7.$$

Therefore
$$d(\Lambda) = \tfrac{1}{3}(Q_3, Q_3') = \frac{1}{3}\left|\frac{x_3}{x_3'}-\frac{x_3}{x_3'}\right| \geqslant \tfrac{1}{3}\sqrt{(7^2-4)} = \sqrt{5},$$

with equality only if R and S lie on L_1; then they must be at P_1 and P_2 respectively, and Λ becomes the lattice Λ_0.

Finally, let $n = 2$. Obviously the centre

$$\tfrac{1}{2}(Q_3-Q_3') = (\xi, \eta),$$

say, of the parallelogram O, $-Q_3'$, Q_3-Q_3', Q_3 is a lattice point. Then for this point, as in the last case, $\xi+\eta \leqslant \sqrt{5}$, and so this centre must lie *on* or *outside* the hyperbola $xy = 1$. Hence

$$\frac{x_3-x_3'}{2}\,\frac{y_3-y_3'}{2} = \frac{x_3-x_3'}{2}\,\frac{-x_3^{-1}+x_3'^{-1}}{2} \geqslant 1$$

or
$$\frac{x_3}{x_3'}+\frac{x_3'}{x_3} \geqslant 4+2 = 6.$$

Therefore
$$d(\Lambda) = \tfrac{1}{2}(Q_3, Q_3') = \frac{1}{2}\left|\frac{x_3}{x_3'}-\frac{x_3'}{x_3}\right| \geqslant \tfrac{1}{2}\sqrt{(6^2-4)} = \sqrt{8} > \sqrt{5},$$

which means that a lattice of this kind cannot be critical.

This completes the proof of Theorem 1, namely, that $\Delta(K) = \sqrt{5}$. In order to obtain also the proof of Theorem 2, i.e. of *case b*, it suffices to show that the critical lattices Λ_0, Λ_2, as obtained in *case c*, are changed by every transformation Ω_t into new lattices which are still K-admissible, and so are critical because their determinant is still $\sqrt{5}$.

Let then $Q_1 = (x_1, y_1)$ on L_1, and $Q_3 = (x_3, y_3)$ and $Q_3' = (x_3', y_3')$ on L_3 be lattice points of Λ_0 or Λ_2 such that $Q_3 = Q_1 + Q_3'$. Then Q_1, Q_3' form a basis, and every lattice point is given by

$$Q = uQ_1 + vQ_3' = (ux_1+vx_3',\ ux_1^{-1}-vx_3'^{-1}) = (x, y),$$

say, where $u, v = 0, \pm 1, \pm 2, \ldots$. Since Q_3 lies on $xy = -1$,

$$x_3y_3 = (x_1+x_3')(y_1+y_3') = (x_1+x_3')\left(\frac{1}{x_1}-\frac{1}{x_3'}\right) = \frac{x_3'}{x_1}-\frac{x_1}{x_3'} = -1,$$

and so, for all lattice points Q,

$$xy = (ux_1+vx_3')\left(\frac{u}{x_1}-\frac{v}{x_3'}\right) = u^2-uv-v^2.$$

Therefore $|xy| \geqslant 1$ for all lattice points different from O. Since a transformation Ω_t does not change the value of xy, Theorem 2 is proved. We see, moreover, that there are no critical lattices in case b, since then Λ would contain a point on L_2 or L_4, and for such a point $|xy| < 1$, contrary to what has just been proved.

Proof of Theorem 3

Since H is contained in, but different from K, there is at least one point R on the boundary L of K which lies outside H. Since the boundary of K is a Jordan curve, not only R, but also a small arc of L containing R, lies outside H. We may therefore assume that R is different from the ten points $\pm P_1$, $\pm P_2$, $\pm(2P_2-P_1)$, $\pm(P_2-P_1)$, $\pm(P_2-2P_1)$ of Λ_0 on L; for otherwise we may replace R by a neighbouring point on L having this property and lying outside H.

By Theorem 2, there exists a critical lattice Λ of K which contains the point R on L. This lattice is also H-admissible. It is, however, not a critical lattice of H. For Λ contains six points on the boundary L of K, and so at most four points on the boundary of H; and so Λ would be a singular lattice of H. Then the tac-line conditions (see the preface) must be satisfied by the four points on the boundary of H. These four points lie also on the boundary of K, and we have shown in the proof of Theorem 1 that the tac-line conditions never hold for points on the boundary of K. Hence Λ is not a critical lattice of H, and so there exist critical lattices of smaller determinant than $\Delta(K)$, as was to be proved.

I wish to express my thanks to Prof. Mordell and Prof. Hardy for their help with the manuscript.

MATHEMATICS DEPARTMENT
MANCHESTER UNIVERSITY

ON LATTICE POINTS IN THE DOMAIN $|xy| \leqslant 1$, $|x+y| \leqslant \sqrt{5}$, AND APPLICATIONS TO ASYMPTOTIC FORMULAE IN LATTICE POINT THEORY (II)

BY K. MAHLER

Communicated by L. J. MORDELL

Received 25 June 1943

I. LATTICE POINTS IN THE DOMAIN $|x|^\alpha + |y|^\alpha \leqslant 1$

THEOREM 4. *Let G be the star domain*

$$|x|^\alpha + |y|^\alpha \leqslant 1,$$

where $\alpha > 0$. Then, when α tends to zero,

$$\Delta(G) = 2^{-2/\alpha}\sqrt{5}\{1+O(\alpha)\}.$$

Proof. The linear substitution

$$x = 2^{-1/\alpha}X, \quad y = 2^{-1/\alpha}Y$$

changes G into the similar domain

$$(G') \qquad |X|^\alpha + |Y|^\alpha \leqslant 2,$$

and so

$$\Delta(G) = 2^{-2/\alpha}\Delta(G').$$

Now

$$|X|^\alpha + |Y|^\alpha = e^{\alpha\log|X|} + e^{\alpha\log|Y|} = 2+\alpha\log|XY| + \rho(X, Y),$$

where, by the mean value theorem of the differential calculus,

$$\rho(X, Y) = \tfrac{1}{2}\alpha^2\{e^{\alpha\theta\log|X|}(\log|X|)^2 + e^{\alpha\theta\log|Y|}(\log|Y|)^2\}$$

with $0<\theta<1$. Hence, for all points on the boundary of G',

$$\log|XY| = -\frac{\rho(X, Y)}{\alpha}, \quad \text{i.e.} \quad |XY| = e^{-\rho(X,Y)/\alpha} \leqslant 1.$$