A THEOREM OF B. SEGRE

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Recent results of B. Segre on lattice points in the star domain (see [2], [3] for the definition and properties of star domains)

$$-a < xy < b \qquad (a > 0, b > 0)$$

contain as a limiting case the following theorem (see [5; Theorems 2, 3]):

Theorem 1. *Let K be the point set*

$$-1 \leq xy < 0.$$

Then every lattice of determinant 1 *has at least one point in K, but a lattice of larger determinant need not have this property.*

This theorem is of interest since K *is not a star domain*; it is moreover nearly trivial that if H is any bounded subset of K, then lattices of arbitrarily small determinant exist which contain no points of H.

In this note, I give a short proof of Theorem 1 based on Mordell's method (for a short account, see [4]), and discuss further the connection with continued fractions.

1. **Proof of Theorem** 1. The parallelogram

$$\Pi: \quad |x + y| \leq 1, \quad |x - y| \leq 2$$

is of area 4; except for the triangle

$$T: \quad x \geq 0, \quad y \geq 0, \quad x + y \leq 1$$

and the triangle $-T$ symmetrical to T in the origin $O = (0, 0)$, Π consists only of points of K.

Let now Λ be any lattice of determinant 1. Then, by Minkowski's theorem on linear forms, at least one point $P_0 = (x_0, y_0) \neq O$ of Λ lies in Π. The assertion is proved if P_0 belongs to K; so let us exclude this case. Then we may assume, without loss of generality, that P_0 lies in T.

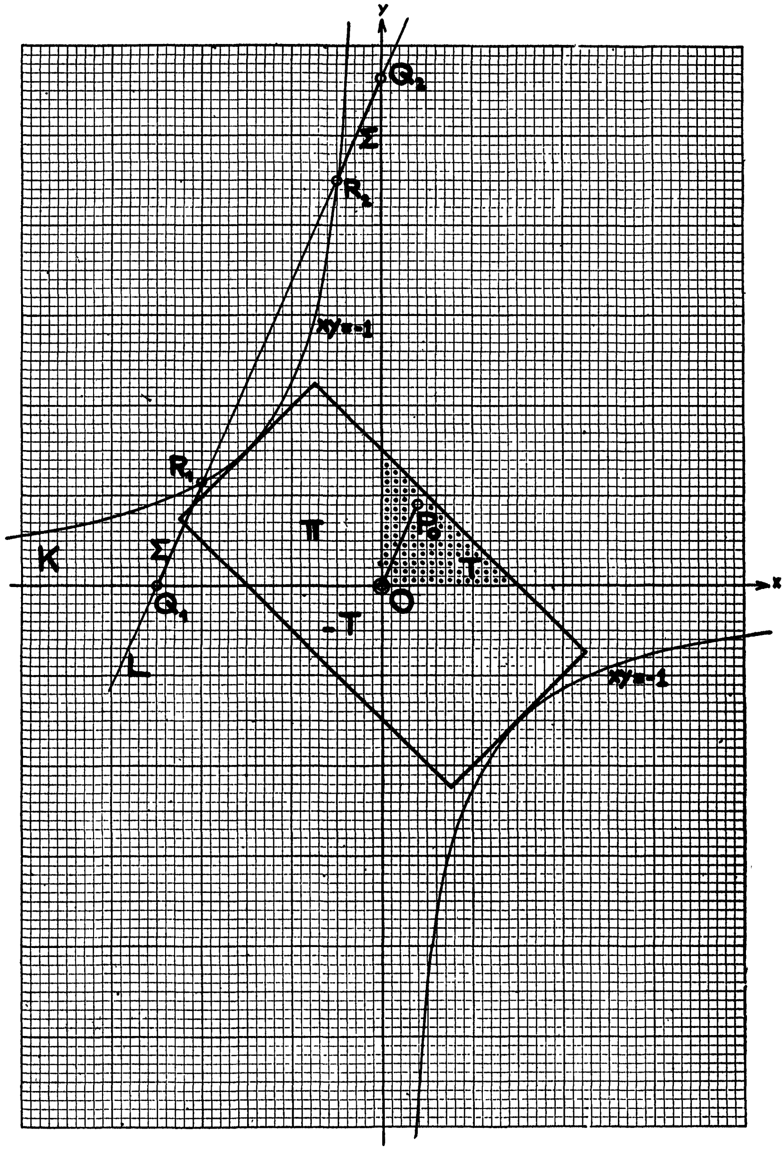
Consider the straight line

$$L: \quad x_0 y - y_0 x = 1.$$

Since Λ is of determinant 1, this line contains an infinity of lattice points, the distance between consecutive points being

$$\overline{OP}_0 = (x_0^2 + y_0^2)^{\frac{1}{2}}.$$

Received December 14, 1944.



Denote by Σ that part of L which belongs to K; Σ consists of one line segment when P_0 lies on the x-axis or y-axis, and otherwise of two segments, abutting the axes of x and y, respectively.

If, firstly, P_0 lies on the y-axis, then $x_0 = 0$ and $0 < y_0 \leq 1$, and L is the straight line $xy_0 + 1 = 0$; L intersects K in the line segment

$$x = -1/y_0\,,\ 0 < y \leq y_0 \text{ of length } y_0 = (x_0^2 + y_0^2)^{\frac{1}{2}} = \overline{OP}_0\,.$$

Hence either there is a point of Λ satisfying $-1 < xy < 0$ and this is an inner point of K, or both points $(-1/y_0, y_0)$ and $(-1/y_0, 0)$ are lattice points. In the second case, Λ has the basis

$$P_0 = (0, y_0), \qquad P_0' = (-1/y_0, 0),$$

since the rectangle of vertices $O, P_0, P_0 + P_0', P_0'$ lies in the region $-1 \leq xy \leq 0$; hence *one* single point of Λ, namely, $P_0 + P_0'$, belongs to K, and an infinity of points of Λ lie on the two axes, i.e., on the closure of K.

If, secondly, P_0 lies on the x-axis, then an analogous result is obtained in the same way.

Let then, thirdly,

$$0 < x_0 \leq 1, \qquad 0 < y_0 \leq 1, \qquad x_0 + y_0 \leq 1,$$

hence

$$0 < \tau = x_0 y_0 \leq \left(\frac{x_0 + y_0}{2}\right)^2 \leq \tfrac{1}{4}.$$

Now L intersects the axes at the two points

$$Q_1 = (-1/y_0, 0), \qquad Q_2 = (0, 1/x_0),$$

and it intersects the boundary $xy = -1$ of K at the two points

$$R_1 = \left(-\frac{1 + (1 - 4\tau)^{\frac{1}{2}}}{2y_0}, \frac{1 - (1 - 4\tau)^{\frac{1}{2}}}{2x_0}\right),$$

$$R_2 = \left(-\frac{1 - (1 - 4\tau)^{\frac{1}{2}}}{2y_0}, \frac{1 + (1 - 4\tau)^{\frac{1}{2}}}{2x_0}\right).$$

Both line segments Q_1R_1 and Q_2R_2 belong to Σ, and they together form Σ. Since

$$R_1 - Q_1 = \left(\frac{1 - (1 - 4\tau)^{\frac{1}{2}}}{2y_0}, \frac{1 - (1 - 4\tau)^{\frac{1}{2}}}{2x_0}\right),$$

$$R_2 - Q_2 = \left(-\frac{1 - (1 - 4\tau)^{\frac{1}{2}}}{2y_0}, -\frac{1 - (1 - 4\tau)^{\frac{1}{2}}}{2x_0}\right),$$

both line segments Q_1R_1 and Q_2R_2 have the same length, namely

$$\left\{\left(\frac{1 - (1 - 4\tau)^{\frac{1}{2}}}{2}\right)^2\left(\frac{1}{x_0^2} + \frac{1}{y_0^2}\right)\right\}^{\frac{1}{2}} = \frac{1 - (1 - 4\tau)^{\frac{1}{2}}}{2\tau}(x_0^2 + y_0^2)^{\frac{1}{2}}.$$

Here the right side is larger than $\overline{OP}_0 = (x_0^2 + y_0^2)^{\frac{1}{2}}$, because $\tau > 0$ and so

$$\frac{1 - (1 - 4\tau)}{2\tau} - 1 = \frac{(1 - 4\tau + 4\tau^2)^{\frac{1}{2}} - (1 - 4\tau)^{\frac{1}{2}}}{2\tau} > 0.$$

Hence both line segments Q_1R_1 and Q_2R_2 contain points of Λ which are *inner* points of K. This proves the first part of the theorem.

If further $d > 1$, then the lattice of basis

$$(-1, 0), \qquad (0, d)$$

and of determinant d contains no point belonging to K. This completes the proof.

2. **The connection with continued fractions.** Consider any lattice Λ without points on the two axes, say of basis $P_0 = (x_0, y_0)$ and $P_0' = (x_0', y_0')$, and of determinant $d(\Lambda) = x_0y_0' - y_0x_0'$; Λ consists therefore of all points $P = (x, y)$, where

$$x = ux_0 + vx_0', \qquad y = uy_0 + vy_0' \qquad (u, v = 0, \mp1, \mp2, \cdots).$$

Then the indefinite quadratic form

$$F_0(u, v) = xy = (ux_0 + vx_0')(uy_0 + vy_0'), = A_0u^2 + 2B_0uv + Cv_0^2$$

say, is of determinant

$$B_0^2 - A_0C_0 = \tfrac{1}{4}d(\Lambda)^2.$$

We assume without loss of generality that A_0 is positive and that $F_0(u, v)$ is a reduced form (see [1; Chapters 3, 4]), hence

$$-\frac{x_0'}{x_0} > 1, \qquad 0 < \frac{y_0'}{y_0} < 1.$$

Let then

$$-\frac{x_0'}{x_0} = g_0 + \frac{1}{g_i +}\frac{1}{g_2 +}\cdots, \qquad \frac{y_0'}{y_0} = \frac{1}{g_{-1} +}\frac{1}{g_{-2} +}\frac{1}{g_{-3} +}\cdots$$

be the regular continued fractions of the two positive numbers $-x_0'/x_0$ and y_0'/y_0; by our hypothesis, both numbers are irrational, and so the continued fractions do not terminate.

As is shown in the theory of reduction of indefinite quadratic forms, all values between $-d(\Lambda)$ and $d(\Lambda)$ assumed by $F_0(u, v)$ belong to the set of numbers (see [1; Chapters 3, 4])

$$(-1)^i\frac{d(\Lambda)}{\theta_i}, \text{ where } \theta_i = \left(g_i + \frac{1}{g_{i+1} +}\cdots\right) + \left(\frac{1}{g_{i-1} +}\frac{1}{g_{i-2} +}\cdots\right)$$
$$(i = 0, \mp1, \mp2, \cdots).$$

Hence, if we denote by Θ the upper bound of θ_i for all positive and negative *odd* indices, then no point of Λ is a point of the point set K considered in §1, if and only if $d(\Lambda)$ satisfies the inequality

$$d(\Lambda) > \Theta.$$

It is, however, evident that

$$\theta_i > 1$$

for all indices i, hence $\Theta > 1$; if, further, $\epsilon > 0$ is arbitrarily small and if

$$g_i = 1 \text{ for all odd indices } i, \qquad g_i > 2/\epsilon \text{ for all even indices } i,$$

then

$$\theta_i < 1 + \epsilon \text{ for all odd indices } i, \qquad \text{hence } \Theta < 1 + \epsilon.$$

Hence every lattice of determinant 1 without points on the axes contains a point of K (and even an infinity of such points); and if $\epsilon > 0$, then there exist lattices without points on the axes and of determinant less than $1 + \epsilon$ which contain no points of K.

Both methods of this note can be extended so as to apply to the more general sets $-a \leq xy \leq b$ and $0 < a \leq xy \leq b$.

References

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