LATTICE POINTS IN TWO-DIMENSIONAL STAR DOMAINS (II)

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In this second part I give, in Chapter II, some general results on symmetrical or unsymmetrical convex domains. In Chapter III, $\Delta(H)$ is determined for any *circle*, *square*, or *triangle* in the (x, y)-plane which contains the origin as an inner point.

CHAPTER II. CONVEX STAR DOMAINS.

18. *A restriction.*

In this chapter we consider only symmetrical or unsymmetrical star domains whose boundaries do *not* include segments of straight lines. Results for the excluded domains can be found by using the method of the first chapter, and are more complicated.

19. *Convex domains symmetrical in the origin.*

We recall that a bounded and closed point set in the (x, y)-plane is called *convex* if with any two points P_1 and P_2 it contains all points

$$tP_1+(1-t)P_2, \quad \text{where} \quad 0 \leqslant t \leqslant 1.$$

The convex set is *symmetrical in a point* Q if with a point P it contains also the point $2Q-P$. We consider only convex domains of which O is an *inner* point, and whose boundary consists of a finite number of analytical arcs; *i.e.* these domains are symmetrical or unsymmetrical simple star domains (§§ 8 and 16).

Let K be a convex domain symmetrical in O. By Theorem 14 and by the restriction in § 18 there are no singular lattices. Therefore every critical lattice of K contains at least three pairs of points $\pm P_1$, $\pm P_2$, $\pm P_3$

on C. The triangle with vertices at P_1, P_2, P_3 lies entirely in K. Hence, by Theorem 11,

$$\text{ind}\,(P_1, P_2) = \text{ind}\,(P_1, P_3) = \text{ind}\,(P_2, P_3) = 1.$$

Hence we may choose P_1 and P_2 as a basis of the lattice, and then, without loss of generality,

$$P_3 = P_1 - P_2.$$

Therefore those admissible lattices are critical which contain three points of this special kind on C and have minimum determinant. This is exactly Minkowski's result†, as it ought to be since our method is derived from his.

The results of the first chapter enable us to determine $\Delta(K)$ for convex domains which are *not* symmetrical in the origin. For a special class of these unsymmetrical convex domains, particularly simple results are obtained in the next paragraph.

20. *Strongly unsymmetrical convex domains.*

Let H be an unsymmetrical convex star domain and let $-H$ be the set of points P for which $-P$ lies in H, and let K be the set of points which belong to at least one of these two domains. We say that H is *strongly unsymmetrical* if the boundaries of H and $-H$ have at most a finite number of points of intersection.

For instance, *every convex star domain H which is symmetrical in a point $Q \neq O$ is strongly unsymmetrical.* For assume that the assertion is false. Then, by Theorem 13, the boundaries of H and $-H$ have an arc A in common, and so also the arc $-A$ symmetrical to A in O. We can find a point P on A which is different from the symmetrical points $-P$ on $-A$ and $2Q-P$ on $2Q-A$, and at which there exists a tangent to the boundary of H. Then H has just one tac-line at each of the three points P, $-P$, and $2Q-P$, and by symmetry these three lines are parallel. This, however, is impossible, since a convex domain has only two tac-lines of given direction.

Let now H be an arbitrary strongly unsymmetrical convex domain, and denote by C_+ and C_- those parts of the boundary C of K which belong to H and $-H$, respectively. The two curves C_+, C_- have only a finite number of common points, say the set Σ^*. The elements of Σ^* can be arranged in pairs P, $-P$ of points; include one of these points in C_+, and the other in C_-.

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† *Diophantische Approximationen* (Leipzig und Berlin, 1907), Kapitel II, § 14.

It is evident from Theorem 14 that the four points of a *singular* lattice of K which lie on the boundary C belong to Σ^*. Since Σ^* has only a finite number of elements, no difficulty arises in finding which of these lead to singular lattices.

Consider now a regular lattice Λ, and let $\pm P_1, \pm P_2, \ldots, \pm P_q$, so that $q \geqslant 3$, be its points on C. Let the notation be such that the points with positive signs belong to C_+, and those with negative signs to C_-. The line segments joining any two of the points $P_1, P_2, \ldots, P_q$ lie entirely in H and so also in K. Hence, by Theorem 11,

$$\text{ind}\,(P_\mu, P_\nu) = 1 \quad (\mu, \nu = 1, 2, \ldots, q;\ \mu \neq \nu). \tag{32}$$

We take P_1 and P_2 as a basis of Λ; by our notation, $-P_1$ and $-P_2$ belong to C_-. From (32), the lattice points $P_3, P_4, \ldots, P_q$ on C are of one of the four types

$$\pm P_1 \pm P_2.$$

Suppose first that $q > 3$, so that at least two of these four points, say P_3 and P_4, belong to C_+. Then $P_3 + P_4 \neq O$, since otherwise one of these points would belong to C_-. Hence, by the form of P_3 and P_4,

$$P_5 = \frac{P_3 + P_4}{2}$$

is also a lattice point different from O. By the convexity of H, P_5 belongs to H and therefore lies on its boundary. The three points P_3, P_4, P_5 on the boundary of H are collinear; hence the line segment joining P_3 and P_4 forms part of this boundary, contrary to the restriction in § 18.

We have, then, $q = 3$, and there is only one further lattice point $P_3 = \pm P_1 \pm P_2$ on C. We may, if necessary, interchange the indices of P_1, P_2, P_3; therefore either

$$P_3 = P_1 - P_2 \quad \text{or} \quad P_3 = -P_1 - P_2.$$

We have thus proved

THEOREM 16. *Let Λ be a regular lattice of a strongly unsymmetrical domain H. Then the set C_+ on the boundary of H contains exactly three lattice points, and these lattice points are of one of the two types,*

$$P_1,\ P_2,\ P_1 - P_2, \quad \textit{or} \quad P_1,\ P_2,\ -P_1 - P_2.$$

CHAPTER III. THE UNSYMMMETRICAL CIRCLE, SQUARE, AND TRIANGLE.

By means of the results so far obtained, I determine the value of $\Delta(H)$ for every circle, square, or isosceles right-angled triangle in the (x, y)-plane which contains the origin O as an inner point. A suitable affine transformation extends these results to arbitrary ellipses, parallelograms, or triangles containing O as inner points.

I am much indebted to Mrs. W. R. Lord for the simple geometrical proof in § 21.

21. *The circle.*

Let

(33) $$0 < a < r$$

and let H be the circle

(34) $$(x-a)^2+y^2 \leqslant r^2$$

of radius r with centre at the point $P:(a, 0)$. By (33), H contains O as an inner point, and is not symmetrical in O.

The domain $-H$ symmetrical to H in O is defined by

(35) $$(x+a)^2+y^2 \leqslant r^2.$$

The boundaries of H and $-H$ intersect at the two points

$$Q': \left(0, \sqrt{(r^2-a^2)}\right) \quad \text{and} \quad Q'': \left(0, -\sqrt{(r^2-a^2)}\right),$$

which are symmetrical in O. Denote by C_+ that arc of the boundary of H for which

(36) $$x \geqslant 0,$$

and by C_- the arc of the boundary of $-H$ symmetrical to C_+ in O.

Since the boundaries of H and $-H$ intersect at two points only, there are no *singular lattices*. Therefore every critical lattice Λ of H contains three points P_1, P_2, P_3 on C_+; the symmetrical points $-P_1$, $-P_2$, $-P_3$ lie on C_-. By (36),

$$P_1+P_2+P_3 \neq O;$$

hence, without loss of generality, by Theorem 16, we may take

(37) $$P_1+P_3 = P_2.$$

The problem now is to find three points P_1, P_2, P_3 on C_+ satisfying (37) and generating a lattice Λ of minimum determinant. This is equivalent to finding a parallelogram Π of minimum area J with vertices at O, P_1, P_2, P_3, where P_1, P_2, P_3 lie on C_+.

The geometrical method of Mrs. Lord proceeds as follows:
Let Q be the centre of Π, that is, the point

$$Q=\frac{P_2}{2}=\frac{P_1+P_3}{2}.$$

Further, denote by u, v, w the distances

$$u=\overline{OQ}=\overline{QP_2},\quad v=\overline{PQ},\quad w=\overline{P_1Q}=\overline{QP_3},$$

and by α the angle OQP. Since QP is perpendicular to P_1P_3, α is the complement of the angle between the diagonals. With our notation,

$$r=\overline{PP_2}.$$

Also by a well-known property of the medians of a triangle

$$2(u^2+v^2)=a^2+r^2. \tag{38}$$

Therefore, by the cosine theorem,

$$r^2=u^2+v^2+2uv\cos\alpha=\frac{a^2+r^2}{2}+2uv\cos\alpha,$$

and

$$\cos\alpha=\frac{r^2-a^2}{4uv}.$$

Hence

$$J=2uw\cos\alpha=\frac{(r^2-a^2)w}{2v}=\frac{r^2-a^2}{2}\sqrt{\left(\frac{r^2}{v^2}-1\right)},$$

since

$$v^2+w^2=r^2.$$

By (38), this expression for J can be written as

$$J=\frac{r^2-a^2}{2}\sqrt{\left(\frac{2r^2}{a^2+r^2-2u^2}-1\right)}.$$

By this formula, J is an increasing function of u, so that its minimum is attained for the smallest possible value of u. Now, if P_2 runs over C_+, then P_1 and P_3 vary on C_+ in the same direction as P_2. Therefore the minimum of J is assumed if either $P_1=Q''$ or $P_3=Q'$. We obtain the same lattice in both cases, namely that with the following four points on C_+,

$$(39)\quad \begin{cases} \left(0,\ \sqrt{(r^2-a^2)}\right), \quad \left(0,\ -\sqrt{(r^2-a^2)}\right), \\ \left(\frac{2a+\sqrt{(a^2+3r^2)}}{2},\ \frac{\sqrt{(r^2-a^2)}}{2}\right), \quad \left(\frac{2a+\sqrt{(a^2+3r^2)}}{2},\ \frac{-\sqrt{(r^2-a^2)}}{2}\right). \end{cases}$$

This lattice Λ_0 is therefore critical; its determinant is

$$d(\Lambda_0) = \frac{\sqrt{(r^2-a^2)}}{2}\{2a+\sqrt{(a^2+3r^2)}\},$$

and there are no further critical lattices. Hence we have

THEOREM 17. *Let a and r satisfy* (33), *and let H be the circle* (34). *Then*

(40) $$\Delta(H) = \frac{\sqrt{(r^2-a^2)}}{2}\{2a+\sqrt{(a^2+3r^2)}\}.$$

There is only one critical lattice Λ_0; this contains the four points (39) *on C_+.*

By continuity, the formula (40) remains true if $a = 0$; in this limiting case we obtain Theorem 3. In the excluded case $a = r$, there are admissible lattices of arbitrary small positive determinant.

22. *The square.*

Let

(41) $$0 < a < r \quad \text{and} \quad 0 < b < r,$$

and let H be the square

(42) $$|x-a| \leqslant r, \quad |y-b| \leqslant r$$

with centre at $P:(a, b)$ and with sides of length $2r$ parallel to the coordinate axes. By (41), H contains O as an inner point, and is not symmetrical in O.

The domain $-H$ symmetrical to H in O is defined by

(43) $$|x+a| \leqslant r \quad \text{and} \quad |y+b| \leqslant r.$$

The boundaries of H and $-H$ intersect at the two points

$$Q':(a-r, r-b) \quad \text{and} \quad Q'':(r-a, b-r)$$

which are symmetrical in O. Denote by C_+ that part of the boundary of H between Q' and Q'' which does not belong to $-H$; and then let C_- be symmetrical to C_+ in O.

Let Λ be a critical lattice. When Λ is singular, then, by Theorem 14, it contains the two points

$$P':(a+r, 0) \quad \text{and} \quad P'':(0, b+r)$$

on C. By Theorem 11, these two points form a basis, and so

$$d(\Lambda) = (r+a)(r+b).$$

I show presently that there are admissible lattices of smaller determinant; hence *there are no singular lattices.*

Let, next, Λ be a regular lattice of H, and let P_1, P_2, P_3 be the three points of Λ on C_+; the symmetrical points $-P_1$, $-P_2$, $-P_3$ lie on C_-. It is again possible to choose the notation so that

$$(44) \qquad P_1+P_3=P_2.$$

The problem is to find three points of this kind such that the area J of the parallelogram Π with vertices at O, P_1, P_2, P_3 is a minimum. The following is a list of all parallelograms which have to be considered:

(*a*) $r-a \leqslant \xi \leqslant r+a$. The vertices of Π are at

$$P_1: (\xi, b-r); \quad P_2: (r+a, 2b); \quad P_3: (r+a-\xi, r+b),$$

and its area is

$$J = J_1(\xi) = 2b\xi+(r+a)(r-b).$$

This area becomes a minimum for $\xi = r-a$, namely

$$J_1 = r^2+(a+b)r-3ab = (r+a)(r+b)-4ab.$$

(*b*) $b-r \leqslant \eta \leqslant 0$. The vertices of Π are at

$$P_1: (a+r, \eta); \quad P_2: (r+a, r+b+\eta); \quad P_3: (0, r+b),$$

and the area is

$$J = J_2 = (r+a)(r+b) > J_1.$$

(*c*) $a-r \leqslant \xi \leqslant 0$. The vertices of Π are at

$$P_1: (r+a, 0); \quad P_2: (r+a+\xi, r+b); \quad P_3: (\xi, r+b),$$

and the area is

$$J = J_3 = (r+a)(r+b) > J_1.$$

(*d*) $r-b \leqslant \eta \leqslant r+b$. The vertices of Π are at

$$P_1: (r+a, r+b-\eta); \quad P_2: (2a, r+b); \quad P_3: (a-r, \eta),$$

and the area is

$$J = J_4(\eta) = 2a\eta+(r-a)(r+b).$$

This area is a minimum for $\eta = r-b$, namely

$$J_4 = J_1.$$

We see from these formulae that there is just one critical lattice Λ_0. This lattice has on C_+ the four points

(45) $$(r-a,\ b-r),\quad (r+a,\ 2b),\quad (2a,\ r+b),\quad (a-r,\ r-b),$$

and its determinant is

$$d(\Lambda_0) = r^2+(a+b)r-3ab.$$

Therefore we have

THEOREM 18. *Let a, b, and r satisfy* (41), *and let H be the square* (42). *Then*

(46) $$\Delta(H) = r^2+(a+b)r-3ab.$$

There is only one critical lattice Λ; this contains the four points (45) *on C_+.*

By continuity, the formula (46) remains true if the signs " $<$ " in (41) are replaced by " $=$ ", provided that the case $a = b = r$ is excluded. In this exceptional case there are admissible lattices of arbitrarily small positive determinant.

23. *The isosceles right-angled triangle.*

Let

(47) $$a > b > 0,\quad 3a+b < 2s,$$

and let H be the isosceles right-angled triangle with vertices at

(48) $$(-a,\ b),\quad (-a+s,\ b+s),\quad (-a+s,\ b-s),$$

and $-H$ the symmetrical triangle with vertices at

$$(a,\ -b),\quad (a-s,\ -b-s),\quad (a-s,\ -b+s).$$

By (47), the origin O is an inner point of the triangle T with vertices at

$$(-a,\ b),\quad \left(-a+\frac{s}{2},\ b-\frac{s}{2}\right),\quad \left(-a+\frac{2s}{3},\ b\right).$$

Here the first point is a vertex of H, the second point lies at the centre of a side of H, and the third point is the centre of gravity of H.

Evidently the parallelogram Π with centre at O and vertices at

$$(a-s,\ -b-s),\quad (s-a,\ -2a-b+s),\quad (-a+s,\ b+s),\quad (a-s,\ 2a+b-s)$$

and of area
$$J = 4(s-a)(a+b)$$

is contained in the star domain K formed by combining H and $-H$. By Minkowski's theorem on linear forms,

$$\Delta(\Pi) = (s-a)(a+b).$$

Since $\Delta(H) = \Delta(K)$, we get, by Theorem 2,

$$\Delta(H) \geqslant (s-a)(a+b).$$

But here the sign of equality holds, since the lattice Λ_0 of basis

$$P_1: (s-a,\ -2a-b+s), \quad P_2: (s-a,\ b+s)$$

and of determinant

$$d(\Lambda_0) = (s-a)(a+b)$$

is easily seen to be H-admissible, and is therefore critical.

Hence we have

THEOREM 19. *Let a, b, and s satisfy* (47), *and let H be the isosceles right-angled triangle with vertices at the points* (48). *Then*

$$\Delta(H) = (s-a)(a+b). \tag{49}$$

By continuity, the formula (49) remains valid if one or more of the signs "$<$" and "$>$" in (47) are replaced by "$=$", provided only that the case $a = b = 0$ is excluded. In this exceptional case there are admissible lattices of arbitrarily small determinant.

24. *The general ellipse, parallelogram, and triangle.*

From the theorems in §§ 21–23 we easily obtain the following more general ones, by applying a suitable affine transformation of the (x, y)-plane. I omit the rather trivial proofs.

THEOREM 20. *Let H be an ellipse of area $J\pi$ which contains O as an inner point. Let the concentric, similar, and similarly situated ellipse through the origin be of area $J_0\pi$. Then*

$$\Delta(H) = \tfrac{1}{2}\left(\sqrt{(J-J_0)}\right)\left(2\sqrt{(J_0)}+\sqrt{(3J+J_0)}\right).$$

THEOREM 21. *Let H be a parallelogram which contains O as an inner point. Let the lines through O parallel to its sides divide H into four parallelograms of areas J_1, J_2, J_3, J_4, where the indices are chosen so that*

$J_1 \leqslant J_2 \leqslant J_3 \leqslant J_4$. *Then*

$$\Delta(H) = J_2 + J_3 - J_1.$$

THEOREM 22. *Let H be a triangle which contains the origin O as an inner point. Let the lines through O parallel to any two of its sides together with the third side form triangles of areas J_1, J_2, J_3, where the notation is such that $J_1 \leqslant J_2 \leqslant J_3$. Then*

$$\Delta(H) = 2\sqrt{(J_2 J_3)}.$$

For symmetrical ellipses and parallelograms, Theorems 20 and 21 reduce to classical results. Theorem 22, however, seems to be the first lattice point property of *triangles* to have been stated.

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