THE THEOREM OF MINKOWSKI-HLAWKA

By Kurt Mahler

Let R_n, where $n \geq 2$, be the n-dimensional Euclidean space of all points $X = (x_1, \cdots, x_n)$ with real coordinates. A symmetrical bounded star body K in R_n is defined as a closed bounded point set containing the origin $O = (0, \cdots, 0)$ as an inner point and bounded by a continuous surface C symmetrical in O which meets every radius vector from O in just one point. A lattice

$$\Lambda: x_h = \sum_{k=1}^{n} a_{hk} u_k \qquad (h = 1, 2, \cdots, n; u_1, \cdots, u_n = 0, \mp 1, \mp 2, \cdots)$$

of determinant

$$d(\Lambda) = \left| \, | a_{hk} |_{h,k=1,2,\ldots,n} \right|$$

is called K-admissible if no point of Λ except O is an inner point of K. Denote by

$$V(K) = \int_K \cdots \int dx_1 \cdots dx_n$$

the volume of K, by $\Delta(K)$ the lower bound of $d(\Lambda)$ extended over all K-admissible lattices, and put

$$Q(K) = \frac{V(K)}{\Delta(K)}.$$

A critical lattice of K is defined as a K-admissible lattice Λ such that $d(\Lambda) = \Delta(K)$.

A theorem due to Minkowski [4; 265, 270, 277], but first proved by E. Hlawka [2; 288–298] and C. L. Siegel [6], states that

$$\text{(a)} \qquad Q(K) \geq 2\zeta(n) \qquad \left(\zeta(n) = \sum_{v=1}^{\infty} v^{-n}\right)$$

for all symmetrical bounded star bodies. It is a difficult problem to decide whether the constant on the right-hand side is the best possible one. In the present note, K is assumed to be a *symmetrical convex body*; under this restriction, the constant $2\zeta(n)$ in (a) will be shown to be replaceable by a larger number.

The method used is quite different from that of Hlawka and Siegel, and depends essentially on the theorem of Brunn and Minkowski on the sections of a convex body. (See [1; §48].)

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1. **Notation.** Let K be any symmetrical convex body in R_n with centre at O, and let $D = \Delta(K)$. Denote

by K_0 the intersection of K with the plane $x_n = 0$, so that K_0 is an $(n-1)$-dimensional convex body symmetrical in O;

by Λ_0 any K_0-admissible $(n-1)$-dimensional lattice in $x_n = 0$;

by $P_h = (x_{h1}, \cdots, x_{h,n-1}, 0)$, where $h = 1, 2, \cdots, n-1$, a basis of Λ_0;

by $d = \| x_{hk} |_{h,k=1,2,\ldots,n-1} |$ the determinant of Λ_0;

by $\xi = (\xi_1, \cdots, \xi_{n-1})$ any point in $(n-1)$-dimensional space R_{n-1};

by W the cube $0 \leq \xi_1 \leq 1, \cdots, 0 \leq \xi_{n-1} \leq 1$ in R_{n-1};

by P_n^* and $P_n^{(\xi)}$ the points

$$P_n^* = \left(0, \cdots, 0, \frac{D}{d}\right), \qquad P_n^{(\xi)} = \xi_1 P_1 + \cdots + \xi_{n-1}P_{n-1} + P_n^*$$

in R_n (Sums of points, or products of points into scalars, have the meaning usual in linear algebra or vector analysis.);

by $\Lambda^{(\xi)}$ the lattice in R_n of basis

$$P_1, P_2, \cdots, P_{n-1}, P_n^{(\xi)}$$

and so of determinant

$$d(\Lambda^{(\xi)}) = d \times \frac{D}{d} = \Delta(K);$$

hence this lattice is either critical, or not K-admissible;

by K_v for $v = 1, 2, 3, \cdots$ the intersection of K with the plane

$$x_n = v\frac{D}{d};$$

since K is bounded, K_v is the null set for all sufficiently large v;

by κ_v the $(n-1)$-dimensional volume

$$\kappa_v = \int_{K_v} \cdots \int dx_1 \cdots dx_{n-1}$$

of K_v; hence

$$\kappa_v = 0$$

when v is sufficiently large;

by c_n any lower bound of $Q(K)$ for all symmetrical convex bodies K in R_n.

2. **The main lemma.** Let $X = (x_1, \cdots, x_{n-1}, vD/d)$ describe the set K_v, and define the point $\xi = (\xi_1, \cdots, \xi_{n-1})$ in R_{n-1} by the equation

$$X = vP_n^{(\xi)} = v(\xi_1 P_1 + \cdots + \xi_{n-1}P_{n-1} + P_n^*),$$

so that

$$x_k = v \sum_{h=1}^{n-1} \xi_h x_{hk} \qquad (k = 1, 2, \cdots, n-1).$$

Then ξ describes a certain set in R_{n-1}, L_v say, and this set is of volume

$$\lambda_v = \int_{L_v} \cdots \int d\xi_1 \cdots d\xi_{n-1} = \frac{\kappa_v}{dv^{n-1}}, \tag{1}$$

since the linear equations connecting the x's with the ξ's are of determinant $v^{n-1}d$.

Next let M_v be the set of all points $\eta = (\eta_1, \cdots, \eta_{n-1})$ in the cube W for which there exist $n - 1$ integers $u_1, \cdots, u_{n-1}$ such that the point

$$X = u_1P_1 + \cdots + u_{n-1}P_{n-1} + vP_n^{(\eta)}$$
$$= (u_1 + v\eta_1)P_1 + \cdots + (u_{n-1} + v\eta_{n-1})P_{n-1} + vP_n^*$$

lies in K_v, and let

$$\mu_v = \int_{M_v} \cdots \int d\eta_1 \cdots d\eta_{n-1}$$

be the volume of this set. Evidently η belongs to M_v if, and only if, the point ξ defined by

$$v\xi_1 = u_1 + v\eta_1, \qquad \cdots, \qquad v\xi_{n-1} = u_{n-1} + v\eta_{n-1}$$

is a point of L_v. Since η lies in W, these equations imply that

$$0 \le v\xi_1 - u_1 < v, \qquad \cdots, \qquad 0 \le v\xi_{n-1} - u_{n-1} < v,$$

and so, for any given point ξ of L_v, each of the integers $u_1, \cdots, u_{n-1}$ has just v possible values. Hence to every point ξ of L_v correspond at most v^{n-1} points η of M_v, obtained by as many translations from ξ. Therefore

$$\mu_v \le v^{n-1}\lambda_v,$$

whence from (1),

$$\mu_v \le \frac{\kappa_v}{d}. \tag{2}$$

LEMMA 1. *The volumes κ_v satisfy the inequality*

$$\sum_{v=1}^{\infty} \kappa_v \ge d.$$

Proof. Let η be any point of W. The lattice $\Lambda^{(\eta)}$ is either *critical* or *not K-admissible.* In the first case, there exist n independent points of $\Lambda^{(\eta)}$ on the boundary of K; in the second case, at least one point of $\Lambda^{(\eta)}$ different from O

is an *inner* point of K and so cannot belong to Λ_0 since Λ_0 is K_0 -admissible. Hence $\Lambda^{(\eta)}$ contains in both cases a point

$$X = u_1P_1 + \cdots + u_{n-1}P_{n-1} + u_nP_n^{(\eta)}$$

of K *not in the plane* $x_n = 0$. Since K is symmetrical, $-X$ also belongs to K; therefore, without loss of generality, the coefficient u_n, or v say, is *positive.* This means that X belongs to K_v, hence η to M_v. The cube W of unit volume is therefore covered completely by the sets $M_1, M_2, M_3, \cdots$, whence

$$\sum_{v=1}^{\infty} \mu_v \geq 1.$$

The assertion follows now from (2). (For Lemma 1 and its proof, see [3]. It was used there for proving a slightly less exact result than the theorem of Minkowski-Hlawka.)

3. **A value for** c_2. In two dimensions, the theorem of Minkowski-Hlawka gives

$$Q(K) \geq c_2, \qquad \text{where} \qquad c_2 = 2\zeta(2) = \frac{\pi^2}{3} = 3.28 \cdots .$$

We show the following better result:

LEMMA 2. *If K is a symmetrical convex region in the plane, then*

$$Q(K) \geq c_2, \qquad \textit{where} \qquad c_2 = 12^{\frac{1}{2}} = 3.46 \cdots .$$

Proof. Assume first that the boundary C of K does not contain any line segments, and choose an arbitrary critical lattice Λ of K. Then this lattice has three points P_1, P_3, P_5 on C such that $P_1 + P_5 = P_3$, and any two of these points form a basis of Λ. Since, if necessary, a suitable affine transformation may be applied, we may assume that P_1, P_3, P_5 are the points

$$P_1 = (1, 0), \qquad P_3 = \left(\frac{1}{2}, \frac{3^{\frac{1}{2}}}{2}\right), \qquad P_5 = \left(-\frac{1}{2}, \frac{3^{\frac{1}{2}}}{2}\right),$$

and that therefore

$$d(\Lambda) = \Delta(K) = \frac{3^{\frac{1}{2}}}{2}.$$

Let

$$P_4 = (0, \xi)$$

be the point where the x_2 -axis intersects C. By the hypothesis about C, P_4 is an inner point of the triangle with vertices at P_3, $P^* = (0, 3^{\frac{1}{2}})$, and P_5, and so

$$\frac{3^{\frac{1}{2}}}{2} < \xi < 3^{\frac{1}{2}},$$

whence

$$\frac{1}{2} < \frac{3^{\frac{1}{2}}}{2\xi} < 1.$$

Therefore the line

$$x = \frac{3^{\frac{1}{2}}}{2\xi}$$

separates P_1 and P_3, and so intersects C in a unique point

$$P_2 = \left(\frac{3^{\frac{1}{2}}}{2\xi}, \eta\right) \qquad \text{with} \qquad 0 < \eta < \frac{3^{\frac{1}{2}}}{2}$$

between P_1 and P_3. Denote by P_6 the point

$$P_6 = P_4 - P_2 = \left(-\frac{3^{\frac{1}{2}}}{2\xi}, \xi - \eta\right);$$

evidently $\xi - \eta > 0$. The lattice Λ^0 of basis P_2, P_4 is of determinant

$$d(\Lambda^*) = \begin{vmatrix} \frac{3^{\frac{1}{2}}}{2\xi} & \eta \\ 0 & \xi \end{vmatrix} = \frac{3^{\frac{1}{2}}}{2} = \Delta(K);$$

it is thus *either critical or not K-admissible.* Hence P_6 either lies *on* C or is an *inner* point of K.

Therefore the 12-sided polygon with vertices at

$$P_1 = (1, 0), P_2 = \left(\frac{3^{\frac{1}{2}}}{2\xi}, \eta\right), P_3 = \left(\frac{1}{2}, \frac{3^{\frac{1}{2}}}{2}\right), P_4 = (0, \xi), P_5 = \left(-\frac{1}{2}, \frac{3^{\frac{1}{2}}}{2}\right),$$

$$P_6 = \left(-\frac{3^{\frac{1}{2}}}{2\xi}, \xi - \eta\right), -P_1, -P_2, -P_3, -P_4, -P_5, -P_6$$

is contained in K. This polygon is of area

$$\begin{vmatrix} 1 & 0 \\ \frac{3^{\frac{1}{2}}}{2\xi} & \eta \end{vmatrix} + \begin{vmatrix} \frac{3^{\frac{1}{2}}}{2\xi} & \eta \\ \frac{1}{2} & \frac{3^{\frac{1}{2}}}{2} \end{vmatrix} + \begin{vmatrix} \frac{1}{2} & \frac{3^{\frac{1}{2}}}{2} \\ 0 & \xi \end{vmatrix} + \begin{vmatrix} 0 & \xi \\ \frac{1}{2} & \frac{3^{\frac{1}{2}}}{2} \end{vmatrix} + \begin{vmatrix} -\frac{1}{2} & \frac{3^{\frac{1}{2}}}{2} \\ -\frac{3^{\frac{1}{2}}}{2\xi} & \xi - \eta \end{vmatrix} + \begin{vmatrix} -\frac{3^{\frac{1}{2}}}{2\xi} & \xi - \eta \\ -1 & 0 \end{vmatrix}$$

$$= \eta + \left(\frac{3}{4\xi} - \frac{\eta}{2}\right) + \frac{\xi}{2} + \frac{\xi}{2} + \left(-\frac{\xi - \eta}{2} + \frac{3}{4\xi}\right) + (\xi - \eta)$$

$$= \frac{3}{2}\left(\frac{1}{\xi} + \xi\right) = 3 + \frac{3(\xi - 1)^2}{2\xi} \geq 3,$$

whence

$$V(K) \geq 3, \qquad Q(K) \geq \frac{3}{3^{\frac{1}{2}}/2} = 12^{\frac{1}{2}},$$

as asserted.

Finally, if C contains line segments, then C can be approximated as nearly as we like by means of symmetrical convex curves with continuous tangent. The assertion follows now from the continuity of $V(K)$ and $\Delta(K)$, hence of $Q(K)$, as functions of K. (The exact value of the lower bound of $Q(K)$ for $n = 2$ will be discussed in more detail in a separate paper.)

4. **Consequences of the theorem of Brunn-Minkowski.** From now on, the symbols K_v and κ_v for the intersection of K with the plane $x_n = vD/d$ and its $(n - 1)$-dimensional volume will be used even if v is not an integer. We further denote by τ_v the quotient

$$\tau_v = \frac{\kappa_v}{\kappa_0};$$

by u and w two numbers such that

$$u < w \qquad \text{and} \qquad \kappa_v > 0 \qquad \text{if} \qquad u < v < w;$$

by $K(u, w)$ the section of K for which

$$uD/d \leq x_n \leq wD/d;$$

and by $V(u, w)$ the volume

$$V(u, w) = \int \cdots \int_{K(u,w)} dx_1 \cdots dx_n$$

of $K(u, w)$.

Lemma 3. *If*

$$v = (1 - t)u + tw \qquad \text{and} \qquad 0 \leq t \leq 1,$$

then

$$\kappa_v^{1/(n-1)} \geq (1 - t)\kappa_u^{1/(n-1)} + t\kappa_w^{1/(n-1)}.$$

This is the theorem of Brunn and Minkowski. (See [1; §48] or [5; §57].)

Lemma 4. *The volume $V(u, w)$ satisfies the inequality*

$$V(u, w) \geq \frac{(w - u)D}{nd} \sum_{h=0}^{n-1} (\kappa_u^h \kappa_w^{n-1-h})^{1/(n-1)}.$$

Proof. Put $v = (1 - t)u + tw$, so that $x_n = vD/d = ((1 - t)u + tw)D/d$.

Then

$$V(u, w) = \int_{uD/d}^{wD/d} \kappa_v \, dx_n = \frac{(w-u)D}{d} \int_0^1 \kappa_v \, dt,$$

whence by Lemma 3,

$$V(u, w) \geq \frac{(w-u)D}{d} \int_0^1 \{(1-t)\kappa_u^{1/(n-1)} + t\kappa_w^{1/(n-1)}\}^{n-1} \, dt.$$

On evaluating the integral, the assertion follows at once.

LEMMA 5. *If* $v \geq 1$, *then*

$$\kappa_v \leq \{v\kappa_1^{1/(n-1)} - (v-1)\kappa_0^{1/(n-1)}\}^{n-1}$$

and

$$\tau_v \leq \{1 - v[1 - \tau_1^{1/(n-1)}]\}^{n-1}.$$

Proof. The first inequality follows from Lemma 3 if u, v, w, t are replaced by 0, 1, v, $1/v$, respectively. The second inequality is obtained from the first one on dividing by κ_0.

LEMMA 6. *If* $v \geq 1$, *then*

$$\tau_v \leq \tau_1^v .$$

Proof. The assertion follows immediately from the last lemma on putting

$$x = 1 - \tau_1^{1/(n-1)}$$

in the well-known inequality

$$1 - vx \leq (1-x)^v.$$

5. **Recursive formulae for** c_n. The evaluation of lower bounds for $Q(K)$ depends on a recursive algorithm. On the assumption that a value for c_{n-1} has already been found, one for c_n is obtained by the following considerations:

Choose the lattice Λ_0 in the plane $x_n = 0$ as a *critical lattice* of K_0; hence, by the induction hypothesis,

$$\kappa_0 \geq c_{n-1}d. \tag{3}$$

Denote further by w the largest positive integer such that

$$\kappa_w > 0. \tag{4}$$

By Lemma 1,

$$\kappa_1 + \kappa_2 + \kappa_3 + \cdots \geq d; \tag{5}$$

hence $w \geq 1$. We distinguish now two cases.

If $w = 1$, then from (5), $\kappa_1 \geq d$. By symmetry, K contains the two congruent convex bodies $K(0, 1)$ and $K(-1, 0)$, and so, by Lemma 4, is of volume

$$V(K) \geq 2V(0, 1) \geq \frac{2D}{nd} \sum_{h=0}^{n-1} (\kappa_0^h \kappa_1^{n-h-1})^{1/(n-1)}$$

$$\geq \frac{2D}{n} \sum_{h=0}^{n-1} c_{n-1}^{h/n-1} = \frac{2D}{n} \frac{c_{n-1}^{n/(n-1)} - 1}{c_{n-1}^{1/(n-1)} - 1}.$$

Hence in this case,

$$Q(K) \geq \overset{\mathrm{i}}{c_n}, \qquad \text{where} \qquad \overset{\mathrm{i}}{c_n} = \frac{2}{n}\left[\frac{c_{n-1}^{n/(n-1)} - 1}{c_{n-1}^{1/(n-1)} - 1}\right]. \tag{6}$$

Next let $w \geq 2$. Then K contains the two congruent convex bodies $K(0, 2)$ and $K(-2, 0)$, and so

$$V(K) \geq 2[V(0, 1) + V(1, 2)]$$

$$\geq \frac{2D}{nd} \left\{ \sum_{h=0}^{n-1} (\kappa_0^h \kappa_1^{n-h-1})^{1/(n-1)} + \sum_{h=0}^{n-1} (\kappa_1^h \kappa_2^{n-h-1})^{1/(n-1)} \right\}.$$

The right-hand side is decreased on replacing κ_2 by 0 and κ_1 by any lower bound for this number. Such a lower bound is deduced from (5) and Lemma 6 as follows:

$$d \leq \kappa_1 + \kappa_2 + \kappa_3 + \cdots = \kappa_0(\tau_1 + \tau_2 + \tau_3 + \cdots) \leq \kappa_0(\tau_1 + \tau_1^2 + \tau_1^3 + \cdots)$$

$$= \frac{\kappa_0 \tau_1}{1 - \tau_1},$$

whence from (3),

$$\tau_1 \geq \frac{d}{\kappa_0 + d}, \qquad \kappa_1 \geq \frac{d\kappa_0}{\kappa_0 + d} \geq \frac{c_{n-1}}{c_{n-1} + 1} d.$$

Hence, on substituting in the inequality above,

$$V(K) \geq \frac{2D}{n} \left\{ \sum_{h=0}^{n-1} \left[c_{n-1}^h \left(\frac{c_{n-1}}{c_{n-1} + 1} \right)^{n-h-1} \right]^{1/(n-1)} + \frac{c_{n-1}}{c_{n-1} + 1} \right\},$$

and so in this case,

$$Q(K) \geq \overset{\mathrm{ii}}{c_n}, \qquad \text{where} \qquad \overset{\mathrm{ii}}{c_n} = \frac{2c_{n-1}}{n(c_{n-1} + 1)} \left\{ \frac{(c_{n-1} + 1)^{n/(n-1)} - 1}{(c_{n-1} + 1)^{1/(n-1)} - 1} + 1 \right\}. \tag{7}$$

The results just found may be formulated as

LEMMA 7. *Put*

$$c_n = \min(\overset{\mathrm{i}}{c_n}, \overset{\mathrm{ii}}{c_n}),$$

where c_n^{i} *and* c_n^{ii} *are defined in* (6) *and* (7); *then*

$$Q(K) \geq c_n$$

for all symmetrical convex bodies in R_n .

6. **The numerical evaluation of** c_n . If, in applying the last lemma, any one of the two constants c_n^{i} and c_n^{ii} is decreased, then c_n does not increase and so remains a lower bound for $Q(K)$. It is therefore permitted to carry out all numerical calculations to only *three places* after the decimal point.

Starting with the value

$$c_2 = 3.464 < 12^{\frac{1}{2}}$$

given by Lemma 2, this remark leads to the following constants:

$$c_3 = 4.216, c_4 = 4.721, c_5 = 5.028, c_6 = 5.187, c_7 = 5.222, c_8 = 5.187,$$

$$c_9 = 5.116, c_{10} = 5.031, c_{11} = 4.942, c_{12} = 4.857, c_{13} = 4.779, c_{14} = 4{,}709,$$

$$c_{15} = 4.646, c_{16} = 4.590, c_{17} = 4.551, c_{18} = 4.505, c_{19} = 4{,}464. c_{20} = 4.428.$$

From the computation,

$$c_n = \begin{cases} c_n^{\text{i}} & \text{for} \quad n \leq 5, \\ c_n^{\text{ii}} & \text{for} \quad 6 \leq n \leq 20, \end{cases}$$

and c_n decreases from $n = 8$ onwards.

7. **The final result.** It still remains to find a lower bound for c_n as n tends to infinity. From their definitions, c_n^{i} and c_n^{ii}, and so also c_n , are greater than

$$c_n^{\text{iii}} = \frac{2c_{n-1}}{n(c_{n-1} + 1)} \frac{(c_{n-1} + 1)^{n/(n-1)} - 1}{(c_{n-1} + 1)^{1/(n-1)} - 1} = \frac{2c_{n-1}}{n} \sum_{h=0}^{n-1} \left(\frac{1}{c_{n-1} + 1}\right)^{h/(n-1)}.$$

Further

$$\begin{aligned} \sum_{h=0}^{n-1} \left(\frac{1}{c_{n-1} + 1}\right)^{h/(n-1)} &\geq \int_0^n \left(\frac{1}{c_{n-1} + 1}\right)^{x/(n-1)} dx \\ &= \frac{(n-1)\{1 - (c_{n-1} + 1)^{-[n/(n-1)]}\}}{\log (c_{n-1} + 1)} \\ &\geq \frac{(n-1)\{1 - (c_{n-1} + 1)^{-1}\}}{\log (c_{n-1} + 1)}, \end{aligned}$$

whence

$$c_n^{\text{iii}} \geq \frac{2(n-1)}{n} \frac{c_{n-1}\{1 - (c_{n-1}+1)^{-1}\}}{\log (c_{n-1}+1)}$$

or

$$c_n^{\text{iii}} \geq \left(2 - \frac{2}{n}\right) \frac{c_{n-1}^2}{(c_{n-1}+1) \log (c_{n-1}+1)}.$$

Write for shortness,

$$\varphi(x) = \frac{19}{10} \frac{x^2}{(x+1) \log (x+1)}.$$

Then by this inequality

$$c_n^{\text{iii}} \geq \varphi(c_{n-1}) \qquad \text{for } n \geq 20,$$

and so even more,

$$(8) \qquad c_n \geq \varphi(c_{n-1}) \qquad \text{for } n \geq 20.$$

Put

$$\psi(x) = (x+2) \log (x+1) - x,$$

so that

$$\varphi'(x) = \frac{19}{10} \frac{x\psi(x)}{\{(x+1) \log (x+1)\}^2}.$$

Since

$$\psi(0) = 0, \qquad \psi'(x) = \log (x+1) + \frac{1}{x+1} > 0 \qquad \text{for } x \geq 0,$$

evidently

$$\psi(x) > 0 \qquad \text{for } x > 0$$

and therefore

$$\varphi'(x) > 0 \qquad \text{for } x > 0.$$

Hence $\varphi(x)$ *is a steadily increasing function of* x.

A simple discussion shows that the equation $\varphi(x) = x$ has the positive root, $x = 3.296 \cdots$; therefore finally,

$$(9) \qquad \varphi(x) > X \quad \text{if} \quad x > X.$$

Since, by the table of the last paragraph, $c_n > X$ for $n \leq 20$, we finally conclude from (8) and (9) by complete induction that $c_n > X$ for all n. Hence the following result has been proved:

THEOREM. *There exists a positive constant a (which may be chosen as* $a = 1/6$*) such that*

$$Q(K) \geq 2\zeta(n) + a$$

for every dimension $n \geq 2$*, and for every symmetrical convex body K in* R_n*. Hence the Theorem of Minkowski-Hlawka does not give the best possible result for such bodies.*

It seems highly probably that the true lower bound of $Q(K)$ tends rapidly to infinity with n. A pointer in this direction is given by the following values for spheres S_n in R_n:

$$Q(S_2) = 3.627 \cdots, \quad Q(S_3) = 5.923 \cdots, \quad Q(S_4) = 9.869 \cdots,$$

$$Q(S_5) = 14.888 \cdots, \quad Q(S_6) = 23.870 \cdots,$$

$$Q(S_7) = 37.798 \cdots, \quad Q(S_8) = 64.940 \cdots.$$

On the other hand, the true lower bound for $Q(K)$, in n dimensions, is almost certainly *not* assumed for S_n, but has a smaller value, and I have in fact proved this when $n = 2$.

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