# ON THE MINIMUM OF A PAIR OF POSITIVE DEFINITE HERMITEAN FORMS

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*Introduction.* Let

$$f_l(x, y) = a_l x\bar{x} + b_l \bar{x} y + \bar{b}_l x\bar{y} + c_l y\bar{y} \quad (l = 1, 2)$$

be a pair of positive definite Hermitean forms of determinants

$$a_1 c_1 - b_1 \bar{b}_1 = a_2 c_2 - b_2 \bar{b}_2 = 1$$

and of simultaneous invariant

$$j = a_1 c_2 - b_1 \bar{b}_2 - \bar{b}_1 b_2 + c_1 a_2;$$

evidently $j \geq 2$ with equality only if the two forms are identical. Also denote by $M(f_1, f_2)$ the smallest value of either $f_1(x, y)$ or $f_2(x, y)$ when $x$ and $y$ take all integral values not both zero in the Gaussian field $K(i)$. The lower bound of $M(f_1, f_2)$ extended over all pairs of forms $f_1$, $f_2$ is a function $m(j)$ of the invariant $j$ only, and its evaluation forms the subject of this paper. By means of the geometrical theory of positive definite Hermitean forms an algorithm for the evaluation of $m(j)$ is developed and applied to the computation of $m(j)$ for $2 \leq j \leq 6$. The result is analogous to that for a pair of positive definite quadratic forms considered by one of us [1]), but the method used there was entirely different.

## CHAPTER I

### THE GEOMETRICAL THEORY

§ 1. *The representative of a Hermitean form.* Let $K(i)$ be

[1]) K. MAHLER, Lattice points in two-dimensional star domains (III), Proc. London Math. Soc. (2), **49** (1946), 168—183.

GAUSS's imaginary quadratic field, and J($i$) the ring of all integers in K($i$). When $\alpha$ is an arbitrary complex number, then denote by R($\alpha$) and I($\alpha$) the real and imaginary parts of $\alpha$, and by $\bar{\alpha}$ the conjugate complex number; if $\alpha$ lies in K($i$), then $\bar{\alpha}$ is its conjugate also with respect to this quadratic field.

Let $\Gamma$ be PICARD's group of linear transformations [2])

$$x = \alpha x' + \beta y', \; y = \gamma x' + \delta y', \tag{1}$$

where $\alpha$, $\beta$, $\gamma$, $\delta$ are elements of J($i$) of determinant

$$\alpha\delta - \beta\gamma = 1.$$

Now let

$$f(x, y) = a x\bar{x} + b\bar{x}y + \bar{b}x\bar{y} + c y\bar{y}$$

be a positive definite Hermitean form of determinant

$$ac - b\bar{b} = 1$$

with arbitrary real coefficient $a$, $c$, and arbitrary complex conjugate coefficients $b$, $\bar{b}$. The transformation (1) changes $f$ into a new positive definite Hermitean form

$$f'(x', y') = a'x'\bar{x}' + b'\bar{x}'y' + \bar{b}'x'\bar{y}' + c'y'\bar{y}'$$

of determinant 1; this new form is called *equivalent* to $f$, in symbols,

$$f \sim f'.$$

We say that $f(x, y)$ is a *reduced form* if for $x$, $y$ in J($i$),

$$f(x, y) \geq \begin{cases} a, & \text{when } \; |x| + |y| > 0, \\ c, & \text{when } \qquad\qquad y = 1. \end{cases} \tag{2}$$

*The form $f$ is reduced if and only if*

$$0 < a \leq c, \quad \left| \mathrm{R}\left(\frac{b}{a}\right) \right| \leq \tfrac{1}{2}, \quad \left| \mathrm{I}\left(\frac{b}{a}\right) \right| \leq \tfrac{1}{2}. \tag{3}$$

*To every form $f$, there exist reduced equivalent forms.* In general, there are just *two* such reduced equivalent forms; these are interchanged by the PICARD transformation

$$x = ix', \; y = -iy'. \tag{4}$$

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[1]) See FRICKE-KLEIN, Automorphe Functionen, Bd. 1, Leipzig 1897, 76—93 and 450—497.

Only when at least *one* sign of equality holds in (3) are there more than two reduced forms equivalent to $f$.

Put

$$\zeta = \frac{b}{a}, \ \eta = \frac{1}{a}, \text{ so that } a = \frac{1}{\eta}, \ b = \frac{\xi}{\eta}, \ c = \frac{\xi\bar{\xi} + \eta^2}{\eta}; \quad (5)$$

further denote by

$$\mathcal{P} : (\xi, \ \eta)$$

the point with rectangular coordinates

$$\mathrm{R}(\xi), \ \mathrm{I}(\xi), \ \eta$$

in three-dimensional upper half-space $\mathrm{P} : \eta > 0$. Then $\mathcal{P}$ is called the *representative of* $f$. The third coordinate $\eta$ of $\mathcal{P}$ is named the *height* of $\mathcal{P}$; this height is a positive number, since $f$ is a positive definite form. The relation between a form and its representative is a one-to-one correspondence; we write in symbols,

$$\mathcal{P} \longleftrightarrow f \text{ or } f \longleftrightarrow \mathcal{P}.$$

When $f$ is changed into $f'$ by the Picard transformation (1), then the representative

$$\mathcal{P}' : (\xi', \eta')$$

of the new form is given by

$$\left.\begin{aligned} \xi' &= \frac{\delta\bar{\gamma}(\xi\bar{\xi} + \eta^2) + \delta\bar{\alpha}\xi + \beta\bar{\gamma}\bar{\xi} + \beta\bar{\alpha}}{\gamma\bar{\gamma}(\xi\bar{\xi} + \eta^2) + \gamma\bar{\alpha}\xi + \alpha\bar{\gamma}\bar{\xi} + \alpha\bar{\alpha}}, \\ \eta' &= \frac{\eta}{\gamma\bar{\gamma}(\xi\bar{\xi} + \eta^2) + \gamma\bar{\alpha}\xi + \alpha\bar{\gamma}\bar{\xi} + \alpha\bar{\alpha}}, \end{aligned}\right\} \quad (6)$$

and further

$$\xi'\bar{\xi}' + \eta'^2 = \frac{\delta\bar{\delta}(\xi\bar{\xi} + \eta^2) + \delta\bar{\beta}\xi + \beta\bar{\delta}\bar{\xi} + \beta\bar{\beta}}{\gamma\bar{\gamma}(\xi\bar{\xi} + \eta^2) + \gamma\bar{\alpha}\xi + \alpha\bar{\gamma}\bar{\xi} + \alpha\bar{\alpha}}. \quad (7)$$

It is well known that these formulae define a *conformal point-transformation* of P into itself, which changes spheres into spheres, planes being considered as spheres of infinite radius. In particular, spheres with their centres in the plane $\eta = 0$ are transformed into spheres of the same kind.

By (3) and (5), the form $f$ is reduced if and only if its representative satisfies the inequalities

$$|\,R(\xi)\,| \leq \tfrac{1}{2},\ |\,I(\xi)\,| \leq \tfrac{1}{2},\ \xi\bar{\xi} + \eta^2 \geq 1,\ \eta > 0. \tag{8}$$

We then call $\mathcal{P}$ a reduced point. The relation

$$f \longleftrightarrow \mathcal{P}$$

evidently defines a one-to-one corrspondence between the elements of the set F of all reduced forms, and the elements of the set $\Phi$ of all reduced points. Corresponding to the Picard transformation (4), the set $\Phi$ is transformed into itself by

$$\xi' = -\xi,\ \eta' = \eta. \tag{9}$$

For all points of $\Phi$,

$$\eta \geq \frac{1}{\sqrt{2}}, \tag{10}$$

with equality only at the four *vertices*

$$\xi = \frac{\mp 1 \mp i}{2},\ \eta = \frac{1}{\sqrt{2}} \tag{11}$$

of $\Phi$.

Let

$$M(f) = \min_{\substack{x,\, y \text{ in } J(i) \\ |x| + |y| > 0}} f(x, y) \tag{12}$$

be the *minimum of* $f$ for $x$, $y$ not both zero in $J(i)$. Then

$$M(f) = M(f') \quad \text{if} \quad f \sim f'. \tag{13}$$

When $f$ is a reduced form, then by (2) and (5),

$$M(f) = a = \frac{1}{\eta}, \tag{14}$$

hence by (10),

$$M(f) \leq \sqrt{2}. \tag{15}$$

Here equality holds if and only if the repsentative of $f$

is one of the four vertices (11) of $\Phi$, i.e. if $f$ is one of the four forms[3]) $(\varepsilon, \varepsilon' = \mp 1)$

$$f(x, y) = \sqrt{2}\left\{x\bar{x} + \frac{\varepsilon + \varepsilon' i}{2}\bar{x}y + \frac{\varepsilon - \varepsilon' i}{2}x\bar{y} + y\bar{y}\right\}. \qquad (16)$$

By (13), the inequality (15) remains valid for non-reduced forms, for there are reduced forms equivalent to any given form.

§ 2. *The problem.* From now on, we consider a system of *two* positive definite Hermitean forms

$$f_l(x, y) = a_l x\bar{x} + b_l \bar{x}y + \bar{b}_l x\bar{y} + c_l y\bar{y} \quad (l = 1, 2) \qquad (17)$$

of determinants

$$a_1c_1 - b_1\bar{b}_1 = a_2c_2 - b_2\bar{b}_2 = 1$$

and of *simultaneous invariant*

$$j = a_1c_2 - b_1\bar{b}_2 - \bar{b}_1b_2 + c_1a_2; \qquad (18)$$

say, for shortness, *a pair of invariant* $j$.

On denoting by

$$\mathcal{P}_l(\xi_l, \eta_l) \text{ with } \eta_l > 0$$

the representatives of these forms, $j$ can be written as

$$j = \frac{(\xi_1 - \xi_2)(\bar{\xi}_1 - \bar{\xi}_2) + (\eta_1 - \eta_2)^2}{\eta_1\eta_2} + 2. \qquad (19)$$

Hence

$$j \geq 2, \qquad (20)$$

with equality if and only if $\mathcal{P}_1$ and $\mathcal{P}_2$ coincide, i.e. $f_1$ and $f_2$ are identical.

If the *same* Picard transformation (1) is applied to both forms of the pair $f_1(x, y)$, $f_2(x, y)$ of invariant $j$, then a new

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[3]) These four forms are equivalent, and are interchanged by the group of four Picard transformations,

$$x = x',\ y = y';\ x = ix',\ y = -iy';\ x = y',\ y = -x';\ x = iy',\ y = ix'.$$

pair $f'_1(x', y')$, $f'_2(x', y')$ of invariant $j$ is obtained. We call this new pair equivalent to the old one, and write

$$(f_1, f_2) \sim (f'_1, f'_2).$$

We further say that the pair $f_1$, $f_2$ of invariant $j$ is *reduced* if

(a) $\qquad f_2$ *is a reduced form*;

(b) $$\mathrm{M}(f_1) \geq \mathrm{M}(f_2).$$

Since every single form can be reduced, there always exists a reduced pair $f'_1$, $f'_2$ equivalent either to $f_1$, $f_2$, or to $f_2$, $f_1$; in the special case that $\mathrm{M}(f_1) = \mathrm{M}(f_2)$, there exist reduced pairs of both kinds.

Put

$$\mathrm{M}(f_1, f_2) = \min\ (\mathrm{M}(f_1),\ \mathrm{M}(f_2)), \tag{21}$$

so that

$$\mathrm{M}(f_1, f_2) = \mathrm{M}(f_2, f_1), \tag{22}$$

$$\mathrm{M}(f_1, f_2) = \mathrm{M}(f'_1, f'_2) \quad \text{if} \quad (f_1, f_2) \sim (f'_1, f'_2). \tag{23}$$

By (15), for every pair of invariant $j$,

$$\mathrm{M}(f_1, f_2) \leq \sqrt{2}. \tag{24}$$

Hence the *smallest upper bound*

$$m(j) = u.b.\mathrm{M}(f_1, f_2) \tag{25}$$

extended over all pairs of invariant $j$, exists; it is a function of $j$ only, and it satisfies the inequality

$$m(j) \leq \sqrt{2}. \tag{26}$$

A second inequality for $m(j)$,

$$m(j) \geq 1, \tag{27}$$

is an immediate consequence of

Theorem 1. *For every value of $j \geq 2$, there exists a pair $f_1$, $f_2$ of invariant $j$ such that*

$$\mathrm{M}(f_1, f_2) = 1.$$

P r o o f. The two forms

$$f_1(x, y) = (x + y\sqrt{j-2})\ (\bar{x} + \bar{y}\sqrt{j-2}) + y\bar{y},$$

$$f_2(x, y) = x\bar{x} + y\bar{y}$$

are positive definite, of determinants 1, and of simultaneous invariant $j$. The second form is reduced, hence $\mathrm{M}(f_2) = 1$. Further for $x, y$ in $\mathrm{J}(i)$,

$$f_1(x, y) \begin{cases} = 1, & \text{when } x = 1,\ y = 0, \\ = x\bar{x} \geq 1, & \text{when } x \neq 0,\ y = 0, \\ \geq y\bar{y} \geq 1, & \text{when } \quad y \neq 0. \end{cases}$$

Hence also $\mathrm{M}(f_1) = 1$, whence $\mathrm{M}(f_1, f_2) = 1$, as was to be proved.

T h e a i m o f t h i s p a p e r i s t o o b t a i n a f i n i t e a l g o r i t h m f o r t h e c o m p u t a t i o n o f $m(j)$.

Since $f_1$ and $f_2$ are identical for $j = 2$, by § 1

$$m(2) = \sqrt{2}. \tag{28}$$

*Therefore let*

$$j > 2$$

*from now on.*

§ 3. *The existence of critical pairs.*

D e f i n i t i o n: *The pair $f_1$, $f_2$ of invariant $j$ is called critical if*

$$\mathrm{M}(f_1, f_2) = m(j).$$

The following existence theorem is fundamental for all that follows:

T h e o r e m 2. *For every value of $j > 2$, there exists at least one critical pair of invariant $j$.*

P r o o f. By the definition of $m(j)$, there exists an infinite sequence of pairs

$$f_1^{(k)}(x, y),\ f_2^{(k)}(x, y) \qquad (k = 1, 2, 3, \ldots.) \tag{29}$$

of invariant $j$ such that [4])

$$\lim_{k \to \infty} \mathrm{M}(f_1^{(k)},\ f_2^{(k)}) = m(j). \tag{30}$$

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[4]) These pairs of forms need not all be different.

*Without loss of generality, these pairs may be assumed to be reduced.*

Then from (30),

$$\lim_{k\to\infty} \mathrm{M}(f_2^{(k)}) = m(j). \tag{31}$$

Hence, to every $\varepsilon > 0$ there is a positive integer $k_0 = k_0(\varepsilon)$ such that

$$m(j) - \varepsilon < \mathrm{M}(f_2^{(k)}) \leq m(j) \text{ for } k \geq k_0. \tag{32}$$

Denote by

$$\mathcal{P}_1^{(k)} : (\xi_1^{(k)}, \eta_1^{(k)}), \quad \mathcal{P}_2^{(k)} : (\xi_2^{(k)}, \eta_2^{(k)}) \quad (k = 1, 2, 3, \ldots .)$$

the representatives of the forms (29). Then $\mathcal{P}_2^{(k)}$ lies in $\Phi$. Hence by (27) and by the formulae in § 1,

$$| \mathrm{R}(\xi_2^{(k)}) | \leq \tfrac{1}{2}, \ | \mathrm{I}(\xi^{(k)}) | \leq \tfrac{1}{2},$$

$$\frac{1}{\sqrt{2}} \leq \frac{1}{m(j)} \leq \eta_2^{(k)} < \frac{1}{m(j) - \varepsilon} \leq \frac{1}{1 - \varepsilon}. \tag{33}$$

Further

$$j = \frac{(\xi_1^{(k)} - \xi_2^{(k)})\,(\bar{\xi}_1^{(k)} - \bar{\xi}_2^{(k)}) + (\eta_1^{(k)} - \eta_2^{(k)})^2}{\eta_1^{(k)}\,\eta_2^{(k)}} + 2,$$

that is:

$$(\xi_1^{(k)} - \xi_2^{(k)})\,(\bar{\xi}_1^{(k)} - \bar{\xi}_2^{(k)}) + \left(\eta_1^{(k)} - \frac{j}{2}\,\eta_2^{(k)}\right)^2 = \frac{j^2 - 4}{4}\,\eta_2^{(k)2}. \tag{34}$$

From (33) and (34),

$$\eta_2^{(k)} \geq \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{2}\,j},$$

$$\eta_1^{(k)} \geq \frac{(j - \sqrt{j^2 - 4})\,\eta_2^{(k)}}{2} = \frac{2\eta_2^{(k)}}{j + \sqrt{j^2 - 4}} > \frac{\eta_2^{(k)}}{j} \geq \frac{1}{\sqrt{2}\,j}. \tag{35}$$

The formulae (33) to (35) show that for $k \geq k_0$ both $\mathcal{P}_1^{(k)}$ and $\mathcal{P}_2^{(k)}$ lie in a bounded closed set B in P which is independent of $k$. Moreover all points in B are of height not smaller than $1/\sqrt{2}j$.

There exists therefore an infinite sequence of indices

$$k_1 < k_2 < k_3 < \dots .$$

such that the corresponding pairs of representatives

$$\mathcal{P}_1^{(k_\nu)},\ \mathcal{P}_2^{(k_\nu)} \qquad (\nu = 1,\ 2,\ 3, \dots .)$$

tend to limit points

$$\lim_{\nu \to \infty} \mathcal{P}_1^{(k_\nu)} = \mathcal{P}_1 : (\xi_1 \eta_1), \quad \lim_{\nu \to \infty} \mathcal{P}_2^{(k_\nu)} = \mathcal{P}_2 : (\xi_2 \eta_2),$$

where

$$\eta_1 > \frac{1}{\sqrt{2j}}, \quad \eta_2 > \frac{1}{\sqrt{2j}}.$$

Therefore the forms belonging to these limit points are positive definite; they are further of determinant 1 and simultaneous invariant $j$. Also

$$\mathrm{M}(f_1, f_2) = m(j),$$

since the minimum of a positive definite form is a continuous function of its coefficient. This concludes the proof [5]).

§ 4. *The equality property of a critical pair.* Let $f_1(x, y)$, $f_2(x, y)$ be a reduced pair of invariant $j$, and

$$\mathcal{P}_1 : (\xi_1, \eta_1), \quad \mathcal{P}_2 : (\xi_2, \eta_2)$$

the representatives of these forms. Hence $\mathcal{P}_2$ lies in $\Phi$, while $\mathcal{P}_1$ does not necessarily do so. Let therefore

$$\mathcal{P}_{\mathrm{I}} : (\xi_{\mathrm{I}}, \eta_{\mathrm{I}})$$

be the representative of a reduced form equivalent to $f_1(x, y)$.

T h e o r e m 3. *If*

$$\mathrm{M}(f_1) > \mathrm{M}(f_2), \tag{36}$$

*then there exists a pair $f_1^*(x, y)$, $f_2^*(x, y)$ of invariant $j$ such that*

$$\mathrm{M}(f_1^*, f_2^*) > \mathrm{M}(f_1, f_2).$$

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[5]) The theorem remains valid for $j = 2$; there are four critical pairs, namely $f_1 \equiv f_2$ must be one of the forms (16).

Proof. Since $M(f_1) = \frac{1}{\eta_I}$, $M(f_2) = \frac{1}{\eta_2}$, the inequality (36) implies

$$\eta_I < \eta_2. \tag{37}$$

Further, since $\mathcal{P}_I$ lies in $\Phi$,

$$\eta_I \geq \frac{1}{\sqrt{2}}, \text{ hence } \eta_2 > \frac{1}{\sqrt{2}}.$$

Therefore $\mathcal{P}_2$ is not one of the vertices

$$\left(\frac{\mp 1 \mp i}{2}, \frac{1}{\sqrt{2}}\right)$$

of $\Phi$, and so there are points

$$\mathcal{P}_2^* : (\xi_2^*, \eta_2^*)$$

of $\Phi$ arbitrary near to $\mathcal{P}_2$, but of height

$$\eta_2^* < \eta_2.$$

To every such point $\mathcal{P}_2^*$ there further exist points

$$\mathcal{P}_1^* : (\xi_1^*, \eta_1^*)$$

such that

$$\frac{(\xi_1^* - \xi_2^*)\ (\bar{\xi}_1^* - \bar{\xi}_2^*) + (\eta_1^* - \eta_2^*)^2}{\eta_1^* \eta_2^*} + 2 = j,$$

and such that $\mathcal{P}_1^*$ tends to $\mathcal{P}_1$ when $\mathcal{P}_2^*$ tends to $\mathcal{P}_2$. Let

$$f_1^*(x, y) \longleftrightarrow \mathcal{P}_1^*, \quad f_2^*(x, y) \longleftrightarrow \mathcal{P}_2^*,$$

and denote by

$$\mathcal{P}_I^* : (\xi_I^*, \eta_I^*)$$

the representative of a reduced form equivalent to $f_1^*(x, y)$. Then

$$\eta_I^* \to \eta_I \text{ when } \mathcal{P}_2^* \to \mathcal{P}_2 \text{ and so } \mathcal{P}_1^* \to \mathcal{P}_1.$$

Therefore, finally, by (37), for $\mathcal{P}_2^* \to \mathcal{P}_2$,

$$\eta_I^* < \eta_2.$$

that is

$$M(f_1^*, f_2^*) = \min\left(\frac{1}{\eta_1^*}, \frac{1}{\eta_2^*}\right) > \frac{1}{\eta_2} = M(f, f_2),$$

as was to be proved.

Theorem 4. *For a critical pair of forms $f_1(x, y), f_2(x, y)$,*

$$M(f_1) = M(f_2).$$

Proof. Evident from the definition and from Theorem 3.

Theorem 5. *To every critical pair $f_1, f_2$ of invariant $j$, there exist a reduced pair $f_1', f_2'$ such that*

$$(f_1', f_2') \sim (f_1, f_2),$$

*and a reduced pair $f_2'', f_2''$ such that*

$$(f_1'', f_2'') \sim (f_2, f_1).$$

Proof. Evident from the definition and from Theorem 4.

Theorem 5 expresses the *symmetry* of a critical pair in its two elements; on account of this theorem, it is sufficient in the next paragraph to prove the assertions always only for one form, say for the form $f_2$.

From now on, the notation remains the same as in the proof of Theorem 3.

§ 5. *The boundary property of a critical pair.*

Theorem 6. *For a critical pair of invariant $j$, both $\mathcal{P}_1$ and $\mathcal{P}_2$ are boundary points of $\Phi$. Moreover, they lie on that part S of the boundary of $\Phi$, which is defined by*

$$\xi\bar{\xi} + \eta^2 = 1, \ |R(\xi)| \leq \tfrac{1}{2}, \ |I(\xi) \leq \tfrac{1}{2}, \ \eta > 0. \qquad (38)$$

Proof. It suffices to show the assertion for $\mathcal{P}_2$. We apply the indirect method and assume that $\mathcal{P}_2$ lies in $\Phi$, but not on S; from this a contradiction will be obtained.

We first remark that

$$\eta_1 \geq \eta_1, \qquad (39)$$

since

$$\frac{1}{\eta_1} = \mathrm{M}(f_1) \leq f_1(1, 0) = a_1 = \frac{1}{\eta_1}.$$

Hence by Theorem 4,

$$\eta_1 \leq \eta_2.$$

Now

$$j = \frac{(\xi_1 - \xi_2)(\bar{\xi}_1 - \bar{\xi}_2) + (\eta_1 - \eta_2)^2}{\eta_1 \eta_2} + 2,$$

that is

$$(\xi_1 - \xi_2)(\bar{\xi}_1 - \bar{\xi}_2) + \left(\eta_2 - \frac{j}{2}\eta_1\right)^2 = \frac{j^2 - 4}{4}\eta_1^2.$$

Hence $\mathcal{P}_2$ lies on a sphere $\Sigma$ of centre

$$\mathrm{Q}: \left(\xi_1, \ \frac{j}{2}\eta_1\right)$$

and radius

$$\varrho = \frac{\sqrt{j^2 - 4}}{2}\eta_1.$$

The lowest point of this sphere is of height

$$\frac{j - \sqrt{j^2 - 4}}{2}\eta_1 = \frac{2\eta_1}{j + \sqrt{j^2 - 4}} < \eta_1 \leq \eta_2,$$

thus of smaller height than $\mathcal{P}_2$.

If now, firstly, $\mathcal{P}_2$ is an inner point of $\Phi$, then there exist points $\mathcal{P}_2^0 : (\xi_2^0, \eta_2^0)$ on $\Sigma$, lying still in $\Phi$ but of height

$$\eta_2^0 < \eta_2.$$

Let $f_2^0(x, y) \longleftrightarrow \mathcal{P}_2^0$. Then $f_1$, $f_2^0$ form a pair of invariant $j$ such that

$$\mathrm{M}(f_1, f_2^0) = \mathrm{M}(f_1) = \mathrm{M}(f_1, f_2), \text{ but } \mathrm{M}(f_2^0) > \mathrm{M}(f_1).$$

Hence, by Theorem 3, there are two forms $f_1^*$, $f_2^*$ of invariant $j$ for which

$$\mathrm{M}(f_1^*, f_2^*) > \mathrm{M}(f_1, f_2),$$

contrary to the hypothesis.

Assume, secondly, that $\mathcal{P}_2$ lies on the boundary of $\Phi$, but not on S; this means that $\mathcal{P}_2$ lies on that part of the boundary of $\Phi$ which is defined by the formulae

$$\begin{aligned} &\mathrm{R}(\xi) = \mp \tfrac{1}{2},\ |\,\mathrm{I}(\xi)\,| \leq \tfrac{1}{2},\ \ \xi\bar{\xi} + \eta^2 > 1,\ \eta > 0 \\ \text{or} \\ &|\,\mathrm{R}(\xi)\,| \leq \tfrac{1}{2},\quad \mathrm{I}(\xi) = \mp \tfrac{1}{2},\ \xi\bar{\xi} + \eta^2 > 1,\ \eta > 0. \end{aligned} \tag{40}$$

The sphere $\Sigma$ passes through $\mathcal{P}_2$ and contains points of smaller height. If at least one of these points of smaller height lies in $\Phi$, then a contradiction is obtained as in the first case.

Assume therefore that all points of $\Sigma$ which have smaller height than $\mathcal{P}_2$, lie outside $\Phi$. Then there exists a transformation

$$\xi' = \xi + \beta,\ \eta' = \eta \qquad (\beta = \mp 1 \text{ or } \beta = \mp i), \tag{41}$$

which

*a*) changes $\mathcal{P}_2$ into a point $\mathcal{P}_2'$ of equal height, also on the boundary (40) of $\Phi$;

*b*) changes $\Sigma$ into a congruent sphere $\Sigma'$ through $\mathcal{P}_2'$, containing at least one point $\mathcal{P}_2^*$ in $\Phi$ arbitrarily near to $\mathcal{P}_2'$ but of smaller height.

Let (41) further transform $\mathcal{P}_1$ into $\mathcal{P}_1'$, and denote by

$$f_1'(x, y) \longleftrightarrow \mathcal{P}_1',\ f_2^*(x, y) \longleftrightarrow \mathcal{P}_2^*$$

the forms of representatives $\mathcal{P}_1'$, $\mathcal{P}_2^*$. Then $f_1'$, $f_2^*$ form a pair of invariant $j$ for which

$$\mathrm{M}(f_1', f_2^*) = \mathrm{M}(f_1') = \mathrm{M}(f_1) = \mathrm{M}(f_1, f_2),\ \text{but}\ \mathrm{M}(f_2^*) > \mathrm{M}(f_1'),$$

and so a contradiction is obtained as in the first case.

## § 6. *The reciprocity theorem.*

Theorem 7. *To every critical pair $f_1$, $f_2$ of invariant $j$ there exists a second critical pair $f_1'$, $f_2'$ of invariant $j$ such that if*

$$\mathcal{P}_1' : (\xi_1', \eta_1'),\qquad \mathcal{P}_2' : (\xi_2', \eta_2'),\qquad \mathcal{P}_{\mathrm{I}}' : (\xi_{\mathrm{I}}', \eta_{\mathrm{I}}')$$

*are the representatives of $f_1'$, $f_2'$ and of a suitable reduced form equivalent to $f_1'$, then*

$$\xi_1' = \frac{\bar{\xi}_1}{\xi_1\bar{\xi}_1 + \eta_1^2}, \quad \eta_1' = \frac{\eta_1}{\xi_1\bar{\xi}_1 + \eta_1^2}, \quad \xi_1'\bar{\xi}_1' + \eta_1'^2 = \frac{1}{\xi_1\bar{\xi}_1 + \eta_1^2}, \tag{42}$$

$$\xi_2' = \bar{\xi}_2, \qquad \eta_2' = \eta_2, \qquad \xi_2'\bar{\xi}_2' + \eta_1'^2 = 1, \tag{43}$$

$$\xi_{\rm I}' = \xi_{\rm I}, \qquad \eta_{\rm I}' = \eta_{\rm I}, \qquad \xi_{\rm I}'\bar{\xi}_{\rm I}' + \eta_{\rm I}'^2 = 1. \tag{44}$$

P r o o f. Put

$$f_1'(x', y') = f_1(y', -x'), \quad f_2'(x', y') = f_2(y', -x'), \tag{45}$$

so that $f_1'$, $f_2'$ is a pair of invariant $j$. Since $(f_1', f_2') \sim (f_1, f_2)$, by (23),

$$\mathrm{M}(f_1', f_2') = \mathrm{M}(f_1, f_2) = m(j);$$

and so $f_1'$, $f_2'$ also form a critical pair. By Theorem 6,

$$\xi_2\bar{\xi}_2 + \eta_2^2 = 1.$$

Hence (42) and (43) follow at once from (6) and (45). Further

$$f_1' \sim f_1, \; f_1 \sim f_{\rm I}, \text{ hence } f_1' \sim f_{\rm I},$$

and so we take $f_{\rm I}$ as the reduced form equivalent to $f_1'$, i.e. $\mathcal{P}_{\rm I}' = \mathcal{P}_{\rm I}$.

R e m a r k s. *a*) The relation between $f_1$, $f_2$ and $f_1'$, $f_2'$ is evidently reciprocal. *b*) If $\mathcal{P}_1$ lies inside the unit sphere, then $\mathcal{P}_1'$ lies outside. *c*) If $\mathcal{P}_{\rm I}$ or $\mathcal{P}_2$ lies on one of the boundary planes

$$\mathrm{R}(\xi) = \mp \tfrac{1}{2} \text{ or } \mathrm{I}(\xi) = \mp \tfrac{1}{2}$$

of $\Phi$, then so does $\mathcal{P}_{\rm I}'$ or $\mathcal{P}_2'$.

§ 7. *The characteristic property of a critical pair.* We now show a property of the critical pairs, by means of which we shall be able to determine these, and so find the value of $m(j)$.

T h e o r e m 8. *For a critical pair $f_1$, $f_2$ of invariant $j$, both $\mathcal{P}_{\rm I}$ and $\mathcal{P}_2$ lie on the circles of intersection of the unit sphere $U$,*

$$\xi\bar{\xi} + \eta^2 = 1,$$

*with the four planes,*

$$R(\xi) = \mp \tfrac{1}{2}, \quad I(\xi) = \mp \tfrac{1}{2}. \tag{46}$$

P r o o f. For reasons of symmetry, it again suffices to prove the assertion for the point $\mathcal{P}_2$: $(\xi_2, \eta_2)$. We apply the indirect method and assume that $\mathcal{P}_2$ lies on none of the planes (46); the same is therefore also true for the point $\mathcal{P}_2'$: $(\bar{\xi}_2, \eta_2)$. From this assumption, we shall derive a contradiction.

By Theorem 7 and the remarks to this theorem, we may further suppose, without loss of generality, that $\mathcal{P}_1$ *is not an inner point of the unit sphere.* For otherwise we only have to replace $f_1, f_2$ by $f_1', f_2'$ as defined in the proof of the last theorem, in order to satisfy this condition.

As we proved in § 5, $\mathcal{P}_2$ lies on the sphere $\Sigma$ with centre at

$$Q : \left(\xi_1, \frac{j}{2}\eta_1\right),$$

and of radius

$$\varrho = \frac{\sqrt{j^2 - 4}\,\eta_1}{2}.$$

Since by $j > 2$,

$$\xi_1\bar{\xi}_1 + \left(\frac{j}{2}\eta_1\right)^2 > \xi_1\bar{\xi}_1 + \eta_1^2 \geq 1,$$

Q lies outside the unit sphere. Denote by C the circle of intersection of the two spheres U and $\Sigma$. Hence $\mathcal{P}_2$ is the lowest point of C. It is even the lowest point of $\Sigma$ inside $\Phi$. For if there were a point $\mathcal{P}_2^0$ of $\Sigma$ in $\Phi$ of smaller height than $\mathcal{P}_2$, then for $f_2^0(x, y) \longleftrightarrow \mathcal{P}_2^0$,

$$M(f_1, f_2^0) = M(f_1, f_2), \quad \text{but} \quad M(f_2^0) > M(f_1)$$

and we should get a contradiction.

The line $\Lambda$ from Q to the centre (0, 0) of U passes through the centre of C; the plane $\Pi$ through $\Lambda$ and $\mathcal{P}_2$ is perpendicular to the plane $\eta = 0$. Hence there is at least one point N on $\Lambda$ such that the line through N and $\mathcal{P}_2$ is perpendicular to $\eta = 0$.

Let K be the cone generated by the lines from N to all points of C. Then Q lies in that part $\Omega$ of the upper half-space P which is bounded by K and the plane through C. For otherwise $\mathcal{P}_2$ would not be the lowest point in $\Phi$ of the circle of intersection of $\Sigma$ with $\Pi$, and so, even more, not the lowest point of $\Sigma$ in $\Phi$.

Hence, if L is the line through Q perpendicular to the plane $\eta = 0$, then this line intersects U in one point R such that R lies on the smaller arc of the greatest circle which connects the centre of C with $\mathcal{P}_2$. In general, R will be of greater height than $\mathcal{P}_2$, since $\mathcal{P}_2$ is the lowest point of C, and so of smaller height than the centre of C. By the definition of Q, $\mathcal{P}_1$ lies on that segment of L which connects Q with R. Hence

$$\eta_1 > \eta_2.$$

This, however, is impossible by (39), since

$$\eta_{\rm I} = \eta_2, \quad \text{therefore } \eta_1 \leq \eta_{\rm I} \leq \eta_2.$$

There is only one exceptional case, in which R need not be of greater height than $\mathcal{P}_2$, but may be of equal height. This happens when $\mathcal{P}_2$ is the highest point (0, 1) of U, and when, at the same time, $\Sigma$ just touches U at the point $\mathcal{P}_2$. The cone K then degenerates into the line $\xi = 0$, and for N may be taken any point of $\xi = 0$ of greater height than $\mathcal{P}_2$. Now R coincides with $\mathcal{P}_2$; therefore $\eta_1 \geq \eta_2$, with equality only for $\mathcal{P}_1 = \mathcal{P}_2$, that is when $f_1$ and $f_2$ are identical. But this case had been excluded.

Hence our original assumption leads in all cases to a contradiction, and so the theorem must be true.

## CHAPTER 2

### The evaluation of $m(j)$

§ 8. *The algebraic formulation of the problem.* By means of Theorem 8, we now obtain a simple rule for finding all reduced critical pairs $f_1(x, y)$, $f_2(x, y)$ of invariant $j$; the non-reduced ones are easily derived from these by applying an arbitrary Picard transformation.

Since the pair $f_1$, $f_2$ is reduced, the second form $f_2$ is reduced; the first form $f_1$ need not be reduced. Then let

$$x = \alpha x' + \beta y',\ y = \gamma x' + \delta y',\quad \alpha\delta - \beta\gamma = 1, \tag{47}$$

or in symbolic form,

$$(x, y) = \Omega(x', y'),\quad \Omega = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix},$$

be a Picard transformation which changes $f_1(x, y)$ into a reduced form

$$f_{\mathrm{I}}(x', y') = a_{\mathrm{I}}x'\bar{x}' + b_{\mathrm{I}}\bar{x}'y' + \bar{b}_{\mathrm{I}}x'\bar{y}' \times c_{\mathrm{I}}y'\bar{y}'. \tag{48}$$

Denote further, as before, by

$$\mathcal{P}_1 : (\xi_1, \eta_1),\quad \mathcal{P}_2 : (\xi_2, \eta_2),\quad \mathcal{P}_{\mathrm{I}} : (\xi_{\mathrm{I}}, \eta_{\mathrm{I}})$$

the representatives of these three forms $f_1$, $f_2$, $f_{\mathrm{I}}$. Then by Theorem 3,

$$\xi_2\bar{\xi}_2 + \eta_2^2 = 1,\ \text{and either } \mathrm{R}(\xi_2) = \mp \tfrac{1}{2} \text{ or } \mathrm{I}(\xi_2) = \mp \tfrac{1}{2};$$

$$\xi_{\mathrm{I}}\bar{\xi}_{\mathrm{I}} + \eta_{\mathrm{I}}^2 = 1,\ \text{and either } \mathrm{R}(\xi_{\mathrm{I}}) = \mp \tfrac{1}{2} \text{ or } \mathrm{I}(\xi_{\mathrm{I}}) = \mp \tfrac{1}{2}.$$

Further by (14) and Theorem 4,

$$\eta_2 = \eta_{\mathrm{I}}, = \eta \tag{49}$$

say, where by (10) and Theorem 1,

$$\frac{1}{\sqrt{2}} \leq \eta \leq 1. \tag{50}$$

The two points $\mathcal{P}_2$ and $\mathcal{P}_{\mathrm{I}}$ are therefore of the form,

$$\mathcal{P}_2 : \left(i^m\left[\zeta + \frac{i\mu}{2}\right], \eta\right),\ \mathcal{P}_{\mathrm{I}} : \left(i^n\left[\zeta + \frac{i\nu}{2}\right], \eta\right) \tag{51}$$

where $\zeta$ is an non-negative number such that

$$(\tfrac{1}{2})^2 + \eta^2 + \zeta^2 = 1,\quad \text{i.e. } \zeta = +\sqrt{\tfrac{3}{4} - \eta^2}, \tag{52}$$

and where further

$m = 0, 1, 2,$ or $3$; $n = 0, 1, 2,$ or $3$; $\mu = \mp 1$; $\nu = \mp 1$

Since $f_1$, $f_2$ is a critical pair,

$$\eta^{-1} = \mathrm{M}(f_1, f_2) = m(j). \tag{53}$$

The single condition (52) does not yet determine $\eta$ and $\zeta$; we need a second equation for this purpose. This equation is contained in the condition that the simultaneous invariant of $f_1$ and $f_2$ has the value $j$. We obtain it in a symmetric form by applying the following method:

The Picard transformation (47) has the inverse,

$$(x', y') = \Omega^{-1}(x, y), \quad \text{where} \quad \Omega^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}. \tag{54}$$

Hence the coefficients of

$$f_1(x, y) = a_1 x\bar{x} + b_1 \bar{x} y + \bar{b}_1 x \bar{y} + c_1 y \bar{y} \tag{55}$$

are given by

$$\begin{aligned} a_1 &= a_{\mathrm{I}}\delta\bar{\delta} - b_{\mathrm{I}}\gamma\bar{\delta} - \bar{b}_{\mathrm{I}}\delta\bar{\gamma} + c_{\mathrm{I}}\gamma\bar{\gamma}, \\ -b_1 &= a_{\mathrm{I}}\beta\bar{\delta} - b_{\mathrm{I}}\alpha\bar{\delta} - \bar{b}_{\mathrm{I}}\beta\bar{\gamma} + c_{\mathrm{I}}\alpha\bar{\gamma}, \\ c_1 &= a_{\mathrm{I}}\beta\bar{\beta} - b_{\mathrm{I}}\alpha\bar{\beta} - \bar{b}_{\mathrm{I}}\beta\bar{\alpha} + c_{\mathrm{I}}\alpha\bar{\alpha}. \end{aligned} \tag{56}$$

On substituting these values in

$$j = a_1 c_2 - b_1 \bar{b}_2 - \bar{b}_1 b_2 + c_1 a_2,$$

we find

$$j = \begin{array}{|c|c|c|c|c} \alpha & \beta & \gamma & \delta & \\ \hline a_2 c_{\mathrm{I}} & -a_2 \bar{b}_{\mathrm{I}} & -b_2 c_{\mathrm{I}} & b_2 \bar{b}_{\mathrm{I}} & \bar{\alpha} \\ \hline -a_2 b_{\mathrm{I}} & a_2 a_{\mathrm{I}} & b_2 b_{\mathrm{I}} & -b_2 a_{\mathrm{I}} & \bar{\beta} \\ \hline -\bar{b}_2 c_{\mathrm{I}} & \bar{b}_2 \bar{b}_{\mathrm{I}} & c_2 c_{\mathrm{I}} & -c_2 \bar{b}_{\mathrm{I}} & \bar{\gamma} \\ \hline \bar{b}_2 b_{\mathrm{I}} & -\bar{b}_2 a_{\mathrm{I}} & -c_2 b_{\mathrm{I}} & c_2 a_{\mathrm{I}} & \bar{\delta} \\ \hline \end{array}\,, \tag{57}$$

where the symbol on the right-hand side stands in an obvious manner for the quarternary Hermitean form

$$a_2 c_{\mathrm{I}} \alpha\bar{\alpha} + a_2 a_{\mathrm{I}} \beta\bar{\beta} + \ldots . - a_2 \bar{b}_{\mathrm{I}} \beta\bar{\alpha} - a_2 b_{\mathrm{I}} \alpha\bar{\beta} + \ldots .$$

By the results of the first chapter

$$a_2 = c_2 = \frac{1}{\eta}, \quad b_2 = \frac{\xi_2}{\eta}; \qquad a_{\mathrm{I}} = c_{\mathrm{I}} = \frac{1}{\eta}\, b, \qquad {}_{\mathrm{I}} = \frac{\xi_{\mathrm{I}}}{\eta}; \tag{58}$$

hence (57) can also be written as

$$\eta^2 j = \begin{array}{|c|c|c|c|c} \alpha & \beta & \gamma & \delta & \\ \hline 1 & -\bar{\xi}_1 & -\xi_2 & \xi_2\bar{\xi}_{\mathrm{I}} & \bar{\alpha} \\ \hline -\xi_{\mathrm{I}} & 1 & \xi_2\xi_{\mathrm{I}} & -\xi_2 & \bar{\beta} \\ \hline -\bar{\xi}_2 & \bar{\xi}_2\bar{\xi}_{\mathrm{I}} & 1 & -\bar{\xi}_{\mathrm{I}} & \bar{\gamma} \\ \hline \bar{\xi}_2\xi_{\mathrm{I}} & -\bar{\xi}_2 & -\xi_1 & 1 & \bar{\delta} \\ \hline \end{array}, = \Phi(\alpha, \beta, \gamma, \delta \mid \xi_2, \xi_{\mathrm{I}}) \tag{59}$$

say. On substituting the values

$$\xi_2 = i^m\left(\zeta + \frac{i\mu}{2}\right), \quad \xi_{\mathrm{I}} = i^n\left(\zeta + \frac{i\nu}{2}\right), \quad \eta^2 = \tfrac{3}{4} - \zeta^2 \tag{60}$$

from (51) and (52) into (59), we obtain a quadratic equation:

$$(\tfrac{3}{4} - \zeta^2)\, j = \Phi\left(\alpha, \beta, \gamma, \delta \,\middle|\, i^m\left(\zeta + \frac{i\mu}{2}\right), \; i^n\left(\zeta + \frac{i\nu}{2}\right)\right) \tag{61}$$

for $\zeta$, which determines $\zeta$ as a function of

$$j, \; \alpha, \; \beta, \; \gamma, \; \delta, \; m, \; n, \; \mu, \; \nu.$$

By (50), this equation has a root $\zeta$ such that

$$0 \leq \zeta \leq \tfrac{1}{2}. \tag{62}$$

When $\zeta$ has thus been found, then $\xi_2$, $\xi_{\mathrm{I}}$, $\eta$ and so the pair $f_1$, $f_2$ of invariant $j$ are determined from (60). In particular,

$$\mathrm{M}(f_1, f_2) = \frac{1}{\eta} = \frac{1}{\sqrt{\frac{3}{4} - \zeta^2}}.$$

This result now leads to the following rule for the determination of all reduced critical pairs $f_1$, $f_2$ of invariant $j$:

Rule: Solve the quadratic equations (61) for all matrices $\Omega = \binom{\alpha\ \beta}{\gamma\ \delta}$ in $\mathrm{J}(i)$ of determinant 1, and for all values of $m, n = 0, 1,$

2, 3; $\mu$, $\nu = \mp 1$. Retain only those equations which have a solution $\zeta$ satisfying (62) [6]. Finally omit all equations except only those in which $\zeta$ assumes the maximum value $\zeta_{max}$. Then the corresponding pairs $f_1$, $f_2$ as defined above, are critical and there are no other critical pairs; further the maximum $m(j)$ is given by the equation

$$m(j) = \frac{1}{\sqrt{\frac{3}{4} - \zeta_{max}^2}}. \tag{63}$$

It will be our aim in the next paragraphs to simplify this rule and to bring it into a practicable form for the computation of $m(j)$.

§ 9. *The extended group $\Gamma^*$.* The *rule* in § 8 can be simplified if $\Gamma$ is replaced by a larger group, which is also due to PICARD.

Denote by $\Gamma^*$ the group of all linear transformations

$$(x, y) = \Omega(x', y'),\ \Omega = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

of determinant

$$| \Omega | = \alpha\delta - \gamma\beta = \iota,$$

where $\iota = \mp 1$ or $\mp i$ is any unit in $K(i)$. Hence $\Gamma$ is a subgroup of $\Gamma^*$ of index 4.

If the form $f(x, y)$ is changed into the new form

$$f'(x', y') = a'x'\bar{x}' + b'\bar{x}'y' + \bar{b}'x'\bar{y}' + c'y'\bar{y}'$$

by the transformation $(x, y) = \Omega(x', y')$ in $\Gamma^*$, then $f$ and $f'$ are called $\Gamma^*$-equivalent; we write in symbols,

$$f \approx f'.$$

By the relation $f \longleftrightarrow P$, $\Gamma^*$ induces in the upper half-space

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[6]) We shall see in the next paragraphs that only a finite number of equations are of this kind.

P a group of transformations. These transformations are given by the same formulae (6), (7), as those of $\Gamma$, except that now $\alpha\delta - \beta\gamma$ may be any unit in $K(i)$.

To every form, there exist *four* $\Gamma^*$-equivalent *reduced* forms; these are interchanged by the group of four elements

$$x = x', \quad y = i^g y' \qquad (g = 0, 1, 2, 3) \quad (64)$$

of $\Gamma^*$. In P, the induced automorphisms of the reduced space take the form,

$$\xi' = i^g \xi, \quad \eta' = \eta \qquad (g = 0, 1, 2, 3). \quad (65)$$

Hence to every form $f$ there exists a $\Gamma^*$-equivalent form $f'$ such that

$$0 \leq R\left(\frac{b'}{a'}\right) \leq \tfrac{1}{2}, \ 0 \leq I\left(\frac{b'}{a'}\right) \leq \tfrac{1}{2}, \ 0 < a' \leq c' \quad (66)$$

The analogous formulae for the representative $\mathcal{P}'$: $(\xi', \eta')$ of $f'$ are

$$0 \leq R(\xi') \leq \tfrac{1}{2}, \ 0 \leq I(\xi') \leq \tfrac{1}{2}, \ \xi'\bar{\xi}' + \eta'^2 \geq 1, \ \eta' > 0. \quad (67)$$

Forms or points satisfying these inequalities are called *strongly reduced*. There is in general just *one* strongly reduced form $\Gamma^*$-equivalent to every given form; if, however, at least *one* equality sign holds in (66), then there is more than one form of this kind.

Theorem 9. *Let* $0 \leq \zeta \leq \frac{1}{2}, \eta > 0, \eta^2 + \zeta^2 = \frac{3}{4}$. *Then the eight reduced points*

$$\left(i^l \left[\zeta + \frac{i\lambda}{2}\right], \eta\right) \quad (l = 0, 1, 2, 3; \lambda = \mp 1)$$

*are* $\Gamma^*$-*equivalent.*

Proof. By the four transformations (65), the eight points are $\Gamma^*$-equivalent to the two points

$$\left(\zeta + \frac{i}{2}, \eta\right), \quad (\tfrac{1}{2} + i\zeta, \eta).$$

Since $(\frac{1}{2} + i\zeta) = 1 + i(\zeta + i/2)$, these two points are also $\Gamma^*$-equivalent.

The determinant of *one* positive definite Hermitean form, and the simultaneous invariant of *two* such forms, are unchanged when transformations in $\Gamma^*$ are applied; also the equations (56) remain valid for transformations $(x, y) = \Omega(x', y')$ in $\Gamma^*$. Hence the other formulae in § 8 also hold under this more general assumption.

We may therefore change the method, so far used, in the following manner:

We assume, in agreement with the Theorems 8 and 9, that the critical pair $f_1, f_2$ of invariant $j$ has been chosen such that the representatives $\mathcal{P}_2, \mathcal{P}_I$ coincide in the same point

$$\mathcal{P}_2 = \mathcal{P}_I = \mathcal{P}: \left(\zeta + \frac{i}{2}, \eta\right), \text{ where } 0 \leq \zeta \leq \tfrac{1}{2}, \eta > 0, \eta^2 + \zeta^2 = \tfrac{3}{4}; \quad (68)$$

on the other hand, we allow the transformation $(x, y) = \Omega(x', y')$ which changes $f_1$ into $f_I$, to be an element of $\Gamma^*$. Then we obtain the new

Rule: Solve the quadratic equations

$$(\tfrac{3}{4} - \zeta^2)\, j = \Phi\left(\alpha, \beta, \gamma, \delta \,\middle|\, \zeta + \frac{i}{2}, \zeta + \frac{i}{2}\right) \quad (69)$$

for all matrices $\Omega = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ in $J(i)$ of determinant $\iota = \mp 1$ or $= \mp i$. Retain only those equations which have a root $\zeta$ in the interval $0 \leq \zeta \leq \frac{1}{2}$, and of these equations omit all except those whose root $\zeta$ assumes the maximum value $\zeta_{max}$. Then the corresponding pairs $f_1, f_2$ of invariant $j$ and obtained from the representatives (68) are critical, and there are no other critical pairs; further the maximum $m(j)$ is given by (63).

§ 10. *Properties of* $\Phi(\alpha, \beta, \gamma, \delta \mid \xi_2, \xi_I)$. In order to discuss the equations (69), it is useful to study the general quarternary Hermitean form

$$\Phi(\alpha, \beta, \gamma, \delta \mid \xi_2, \xi_1) = \begin{array}{|c|c|c|c|c} \alpha & \beta & \gamma & \delta & \\ \hline 1 & -\bar{\xi}_1 & -\xi_2 & \xi_2\bar{\xi}_1 & \bar{\alpha} \\ \hline -\xi_1 & 1 & \xi_2\xi_1 & -\xi_2 & \bar{\beta} \\ \hline -\bar{\xi}_2 & \bar{\xi}_2\bar{\xi}_1 & 1 & -\bar{\xi}_1 & \bar{\gamma} \\ \hline \bar{\xi}_2\xi_1 & -\bar{\xi}_2 & -\xi_1 & 1 & \bar{\delta} \\ \hline \end{array} . \qquad (70)$$

It is not difficult to verify that this form has the following symmetry properties:

Theorem 10. *The function $\Phi(\alpha, \beta, \gamma, \delta \mid \xi_2, \xi_1)$ remains unchanged when its six arguments*

$$\alpha, \quad \beta, \quad \gamma, \quad \delta, \quad \xi_2, \quad \xi_1$$

*are, in this order, replaced by* [7])

| | | | | | | |
|---|---|---|---|---|---|---|
| | $\alpha,$ | $\gamma,$ | $\beta,$ | $\delta,$ | $\bar{\xi}_1,$ | $\bar{\xi}_2;$ |
| *or* | $\beta,$ | $\alpha,$ | $\delta,$ | $\gamma,$ | $\xi_2,$ | $\bar{\xi}_1;$ |
| *or* | $\beta,$ | $\delta,$ | $\alpha,$ | $\gamma,$ | $\xi_1,$ | $\bar{\xi}_2;$ |
| *or* | $\gamma,$ | $\alpha,$ | $\delta,$ | $\beta,$ | $\bar{\xi}_1,$ | $\xi_2;$ |
| *or* | $\gamma,$ | $\delta,$ | $\alpha,$ | $\beta,$ | $\bar{\xi}_2,$ | $\xi_1;$ |
| *or* | $\delta,$ | $\beta,$ | $\gamma,$ | $\alpha,$ | $\xi_1,$ | $\xi_2;$ |
| *or* | $\delta,$ | $\gamma,$ | $\beta,$ | $\alpha,$ | $\bar{\xi}_2,$ | $\bar{\xi}_1.$ |

A further important property of $\Phi$ is given by

Theorem 11. *If $|\xi_2| < 1$, $|\xi_1| < 1$, then $\Phi(\alpha, \beta, \gamma, \delta \mid \xi_2, \xi_1)$ is a positive definite Hermitean form of $\alpha$, $\beta$, $\gamma$, $\delta$.*

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[7]) The variables may be changed in many other ways so as to leave $\Phi$ invariant; we may, for instance, replace

| | | | | | | |
|---|---|---|---|---|---|---|
| | $\alpha,$ | $\beta,$ | $\gamma,$ | $\delta,$ | $\xi_2,$ | $\xi_1;$ |
| by | $\bar{\alpha},$ | $\bar{\beta},$ | $\bar{\gamma},$ | $\bar{\delta},$ | $\bar{\xi}_2,$ | $\bar{\xi}_1;$ |
| or | $i\alpha,$ | $\beta,$ | $\gamma,$ | $-i\delta,$ | $i\xi_2,$ | $-i\xi_1;$ |
| or | $i\alpha,$ | $i\beta,$ | $\gamma,$ | $\delta,$ | $i\xi_2,$ | $\xi_1.$ |

In all these cases, $\alpha\delta - \beta\gamma$ is only multiplied by a unit in $K(i)$.

P r o o f. The assertion follows immediately from the identity,

$$\Phi = |\alpha - \bar{\xi}_1\beta - \xi_2\gamma + \xi_2\bar{\xi}_1\delta|^2 + (1 - \xi_1\bar{\xi}_1)|\beta - \xi_2\delta|^2 +$$
$$+ (1 - \xi_2\bar{\xi}_2)|\gamma - \bar{\xi}_1\delta|^2 + (1 - \xi_2\bar{\xi}_2)(1 - \xi_1\bar{\xi}_1)|\delta|^2. \quad (71)$$

From Theorem 11, we can now deduce that only a finite number of equations (69) is solvable in the interval $0 \leq \zeta \leq \frac{1}{2}$:

T h e o r e m 12. *Assume that the equation*

$$(\tfrac{3}{4} - \zeta^2)\, j = \Phi\left(\alpha, \beta, \gamma, \delta \,\middle|\, \zeta + \frac{i}{2}, \zeta + \frac{i}{2}\right)$$

*has a solution in the interval* $0 \leq \zeta \leq \frac{1}{2}$. Then

$$\max(|\alpha|, |\beta|, |\gamma|, |\delta|) \leq \sqrt{2j}. \quad (72)$$

P r o o f. From (71),

$$\Phi\left(\alpha, \beta, \gamma, \delta \,\middle|\, \zeta + \frac{i}{2}, \zeta + \frac{i}{2}\right) \geq$$
$$\geq \left\{1 - \left(\zeta + \frac{i}{2}\right)\left(\zeta - \frac{i}{2}\right)\right\}^2 |\delta|^2 = (\tfrac{3}{4} - \zeta^2)\, |\delta|^2,$$

hence by Theorem 10,

$$\Phi\left(\alpha, \beta, \gamma, \delta \,|\, \zeta + \frac{i}{2}, \zeta + \frac{i}{2}\right) \geq$$
$$\geq (\tfrac{3}{4} - \zeta^2)^2 \max(|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2). \quad (73)$$

Therefore, if $0 \leq \zeta \leq \frac{1}{2}$, then (69) implies

$$j = \frac{\Phi\left(\alpha, \beta, \gamma, \delta \,|\, \zeta + \frac{i}{2}, \zeta + \frac{i}{2}\right)}{\frac{3}{4} - \zeta^2} \geq$$
$$\geq (\tfrac{3}{4} - \zeta^2) \max(|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2 \geq$$
$$\geq \tfrac{1}{2} \max(|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2)$$

and the assertion follows at once.

By means of Theorem 12, we may now express the rule for the determination of $m(j)$ in the following final form:

Rule H: Solve the quadratic equations

$$(\tfrac{3}{4}-\zeta^2)\, j = \Phi\left(\alpha, \beta, \gamma, \delta \,\middle|\, \zeta+\frac{i}{2}, \zeta+\frac{i}{2}\right)$$

for all matrices $\Omega = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in J($i$) of determinant $\iota = \mp 1$ or $= \mp i$, and with elements satisfying

$$\max\,(|\alpha|, |\beta|, |\gamma|, |\delta|) \leq \sqrt{2j}.$$

Retain only those equations which have a solution $\zeta$ in $0 \leq \zeta \leq \frac{1}{2}$ of maximum value $\zeta_{max}$. Then

$$m(j) = (\tfrac{3}{4} - \zeta_{max}^2)^{-\frac{1}{2}},$$

and all critical pairs are found from their representatives (68).

§ 11. *The numerical value of* $m(j)$. We first show that the equation (69) *does not reduce* to an identity. For otherwise we should have

$$-j = \alpha\bar{\delta} + \bar{\alpha}\delta + \beta\bar{\gamma} + \bar{\beta}\gamma,$$

$$0 = (\beta+\gamma)(\bar{\alpha}+\bar{\delta}) + (\bar{\beta}+\bar{\gamma})(\alpha+\delta) + i(\beta\bar{\gamma} - \bar{\beta}\gamma),$$

and $\frac{3}{4} j = \alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta} +$

$$+\frac{i}{2}\{(\beta-\gamma)(\bar{\alpha}+\bar{\delta}) - (\bar{\beta}-\bar{\gamma})(\alpha+\delta)\} + \tfrac{1}{4}(\alpha\bar{\delta}+\bar{\alpha}\delta-\beta\bar{\gamma}-\bar{\beta}\gamma).$$

Hence

$$3(\alpha\bar{\delta}+\bar{\alpha}\delta+\beta\bar{\gamma}+\bar{\beta}\gamma)+4(\alpha\bar{\alpha}+\beta\bar{\beta}+\gamma\bar{\gamma}+\delta\bar{\delta})+2i(\beta-\gamma)(\bar{\alpha}+\bar{\delta})-$$
$$-2i(\bar{\beta}-\bar{\gamma})(\alpha+\delta)+\alpha\bar{\delta}+\bar{\alpha}\delta-\beta\bar{\gamma}-\bar{\beta}\gamma = 0. \quad (74)$$

Now put

$$\alpha = a_1 + ia_2, \qquad \beta = b_1 + ib_2,$$
$$\gamma = c_1 + ic_2, \qquad \delta = d_1 + id_2, \qquad (75)$$

so that (74) takes the form

$$(a_1 + d_1)^2 + (a_2 + d_2)^2 + \tfrac{1}{2}(a_1 - b_2)^2 + \tfrac{1}{2}(b_2 - d_1)^2 + \\ + (c_1 - a_2)^2 + \tfrac{1}{2}(c_1 - d_2)^2 + \tfrac{1}{2}(a_1 + c_2)^2 + \tfrac{1}{2}(c_2 + d_1)^2 + \\ + \tfrac{1}{2}(a_2 + b_1)^2 + \tfrac{1}{2}(b_1 + d_2)^2 + \tfrac{1}{2}(b_2 + c_2)^2 + \tfrac{1}{2}(b_1 + c_1)^2 + \\ + \tfrac{1}{2}\{b_1^2 + b_2^2 + c_1^2 + c_2^2\} = 0. \qquad (76)$$

Since $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, $c_2$, $d_1$, $d_2$ are all rational integers (hence real), all terms in (76) vanish, and so finally, $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, $c_2$, $d_1$, $d_2$ are all zero, contrary to the condition

$$|\alpha\delta - \beta\gamma| = 1.$$

Further computation according to Rule H is simplified by observing that the function $\Phi(\alpha, \beta, \gamma, \delta \mid \zeta + \frac{1}{2}, \zeta + \frac{1}{2})$ remains unchanged when

$$\alpha, \quad \beta, \quad \gamma, \quad \delta,$$

in this order are replaced by

$$\begin{array}{lcccc} & -\alpha, & -\beta, & -\gamma, & -\delta, \\ \text{or} & i\alpha, & i\beta, & i\gamma, & i\delta, \\ \text{or} & \delta, & \beta, & \gamma, & \alpha, \\ \text{or} & \bar{\alpha}, & \bar{\gamma}, & \bar{\beta}, & \bar{\delta}. \end{array}$$

Also the equation (69) can be written as

$$\psi(\zeta;\ \alpha, \beta, \gamma, \delta) = j, \qquad (77)$$

where

$$\psi(\zeta;\ \alpha, \beta, \gamma, \delta) = \frac{\Phi(\alpha, \beta, \gamma, \delta \mid \zeta + \frac{1}{2}, \zeta + \frac{1}{2})}{(\frac{3}{4} - \zeta^2)}.$$

Using the result of Theorem 11, it is easily seen that $\psi$ is positive for $0 \leq \zeta \leq \frac{1}{2}$.

We consider only the interval $2 \leq j \leq 6$ for $j$. We have to find all matrices $\Omega = \binom{\alpha\ \beta}{\gamma\ \delta}$ in $J(i)$ of determinant $\iota = \mp 1$ or $\iota = \mp i$, where $\max(|\alpha|, |\beta|, |\gamma|, |\delta|) \leq \sqrt{2j}$, for which $\psi$ assumes values between 2 and 6 for suitable values of $\zeta$ satisfying $0 \leq \zeta \leq \frac{1}{2}$. A discussion which is somewhat laborious and in which more than three hundred matrices have to be considered leads to the following table:

TABLE OF ALL FUNCTIONS $\psi$ WHICH REPRESENT $j$ FOR $2 \leq j \leq 6$ (Cont.).

| $\psi(\frac{1}{2})$ | $(3/4 - \zeta^2)\,\psi(\zeta;\alpha,\beta,\gamma,\delta)$ | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | $2 + 2\zeta$ | 0 | $i$ | 1 | 0 | 0 | $i$ | $-1$ | $1-i$ | 1 | $-1+i$ | $-i$ | 1 |
| | | 1 | $-1$ | 0 | $i$ | | | | | | | | |
| | $5/2 + 2\zeta^2$ | 0 | $i$ | $i$ | 1 | 1 | 0 | 0 | 1 | 1 | $2i$ | 0 | 1 |
| | | 1 | $i$ | $-i$ | 2 | | | | | | | | |
| | $7/2 - 2\zeta^2$ | 0 | 1 | $-1$ | 1 | 0 | $i$ | $-i$ | $2+i$ | 1 | $1+i$ | 0 | $-1$ |
| | | 1 | $1-i$ | 0 | $-1$ | 1 | $-1$ | $-i$ | $-1+i$ | 1 | 1 | $-i$ | $-1-i$ |
| | | 1 | $i$ | $1-2i$ | $1+i$ | 1 | $i$ | $-1-2i$ | $1-i$ | | | | |
| | $4 - 2\zeta$ | 0 | $i$ | 1 | 2 | 0 | 1 | $i$ | $1-i$ | 0 | 1 | $-i$ | $2-i$ |
| | | 1 | $i$ | 2 | $i$ | 1 | $1+i$ | $i$ | $-1$ | 1 | $i$ | $1-i$ | $1+2i$ |
| | | 1 | $i$ | $-2i$ | $2-i$ | 1 | 1 | 0 | $i$ | 2 | $i$ | $1-2i$ | $1+i$ |
| | $5 - 2\zeta - 4\zeta^2$ | 1 | 1 | $-1+i$ | $-1$ | 1 | $-1+i$ | 1 | $-1+2i$ | 1 | $1+i$ | $-1-2i$ | $1-2i$ |
| | | $1+i$ | $i$ | $-1$ | $-1-i$ | $1+i$ | $i$ | $1-2i$ | $1-i$ | | | | |
| | $11/2 - 4\zeta - 2\zeta^2$ | 1 | $-1+2i$ | 1 | $2i$ | 1 | 1 | $i$ | $-1+i$ | 1 | $2i$ | $1-i$ | $1+2i$ |
| | | $1+i$ | $2-i$ | $-1$ | $1-2i$ | | | | | | | | |
| | $6 - 6\zeta$ | 0 | 1 | $i$ | $1+i$ | 0 | $i$ | 1 | $2+2i$ | 1 | $i$ | $1+i$ | $-1+2i$ |
| | | $1+i$ | $-1+2i$ | $2-i$ | $2+2i$ | | | | | | | | |
| | $13/2 - 8\zeta + 2\zeta^2$ | 1 | 0 | 2 | 1 | 0 | 1 | 1 | $2+i$ | 1 | $i$ | $2+i$ | $2i$ |
| | | 1 | $i$ | $2-i$ | $2+2i$ | | | | | | | | |
| | $7 - 10\zeta + 4\zeta^2$ | 1 | 1 | $1+i$ | 1 | 1 | $1+i$ | 1 | $1+2i$ | 1 | $1+i$ | $1-2i$ | 3 |
| | | $1+i$ | 1 | $2-i$ | $1-i$ | $1+i$ | $i$ | 3 | $1+i$ | | | | |
| | $15/2 - 12\zeta + 6\zeta^2$ | 1 | 1 | 1 | 2 | 1 | $2+i$ | 1 | $1+i$ | 1 | $1+i$ | 2 | $1-2i$ |
| | | 1 | $1+i$ | $1-i$ | 3 | 2 | $1+2i$ | $1-2i$ | 3 | | | | |
| | $19/2 - 16\zeta + 6\zeta^2$ | 1 | 1 | $2+i$ | $1+i$ | 1 | $1+i$ | $1+i$ | $1+2i$ | 1 | $1+2i$ | 1 | $2+2i$ |
| | | $1+i$ | $1+2i$ | $2-i$ | 3 | $1+i$ | $2+i$ | 1 | $2-i$ | | | | |
| | $10 - 18\zeta + 8\zeta^2$ | 1 | $2+i$ | $1-i$ | 3 | 1 | 1 | 2 | $2+i$ | 1 | $1+i$ | $2+i$ | $1+2i$ |
| | | $1+i$ | $2+i$ | 1 | 2 | 2 | $1+2i$ | $2-i$ | $2+2i$ | 3 | $1+2i$ | $2-2i$ | $2+i$ |
| | $14 - 30\zeta + 16\zeta^2$ | 3 | $2+2i$ | $2-i$ | $2+i$ | $1+i$ | $1+2i$ | $2+i$ | $2+2i$ | | | | |

TABLE OF ALL FUNCTIONS $\psi$ WHICH REPRESENT $j$ FOR $2 \leq j \leq 6$

| $\psi(\frac{1}{2})$ | $(3/4 - \zeta^2)\ \psi(\zeta; \alpha, \beta, \gamma, \delta)$ | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | $3/2 - 2\zeta^2$ | 0 | $i$ | $-i$ | 1 | 1 | 0 | 0 | $-1$ | | | | |
| | $2 - 2\zeta$ | 0 | 1 | $i$ | 0 | 0 | $i$ | 1 | $1+i$ | 1 | $i$ | $1-i$ | 1 |
| | | 1 | 1 | 0 | $-i$ | | | | | | | | |
| | $5/2 - 4\zeta + 2\zeta^2$ | 0 | 1 | 1 | 1 | 1 | $1+i$ | 0 | 1 | 1 | $1+i$ | $1-i$ | 1 |
| | | 1 | $i$ | 1 | $1+i$ | | | | | | | | |
| | $3 - 6\zeta + 4\zeta^2$ | 1 | 1 | $1-i$ | 1 | $1+i$ | $i$ | 1 | $1+i$ | | | | |
| 4 | $3/2 + 2\zeta^2$ | 0 | 1 | 1 | 0 | 1 | $i$ | 0 | 1 | | | | |
| | $2$ | 0 | $i$ | $-1$ | 1 | 0 | $i$ | 1 | 1 | 1 | $i$ | $-i$ | $1+i$ |
| | | 1 | $-1+i$ | 0 | $i$ | | | | | | | | |
| | $5/2 - 2\zeta^2$ | 0 | 1 | $-1$ | 0 | 0 | 1 | $-1$ | $2i$ | 0 | $-1$ | 1 | $1+i$ |
| | | 1 | 1 | 0 | $-1$ | 1 | $i$ | 0 | $-1$ | 1 | $2i$ | $-i$ | 1 |
| | | 1 | $-1$ | $1-i$ | $i$ | 1 | $-1+i$ | 1 | $i$ | $1+i$ | $i$ | $-i$ | $1-i$ |
| | $3 - 4\zeta^2$ | 1 | $-1$ | 1 | $-1+i$ | | | | | | | | |
| | $7/2 - 4\zeta + 2\zeta^2$ | 0 | 1 | 1 | $1+i$ | 1 | 0 | 1 | 1 | 1 | $1+i$ | 1 | $i$ |
| | | 1 | $i$ | $1-i$ | $2+i$ | $1+i$ | $-1+2i$ | 1 | $1+i$ | | | | |
| | $4 - 4\zeta$ | 0 | 1 | $i$ | 1 | 1 | $1+i$ | 0 | $i$ | 1 | 0 | 2 | $i$ |
| | | 1 | $i$ | $1-2i$ | 2 | 1 | $i$ | 1 | $2i$ | 1 | $i$ | $i$ | $-1+i$ |
| | | 1 | $2i$ | $1-i$ | $2+i$ | 1 | $i$ | $2-i$ | $1+i$ | | | | |
| | $5 - 8\zeta + 4\zeta^2$ | 1 | 1 | 1 | $1+i$ | 1 | $1+2i$ | 1 | $1+i$ | 1 | $1+i$ | $1-i$ | $2+i$ |
| | | $1+i$ | $i$ | $2-i$ | $2+i$ | | | | | | | | |
| | $11/2 - 8\zeta + 2\zeta^2$ | 0 | 1 | 1 | 2 | 1 | $1+i$ | $i$ | $i$ | 1 | $i$ | 2 | $1+2i$ |
| | | 1 | $2i$ | 1 | $1+2i$ | 1 | $1+2i$ | $1-i$ | $2+i$ | 2 | $1+2i$ | $1-2i$ | 2 |
| | | 2 | $i$ | $2-i$ | $1+i$ | | | | | | | | |
| | $13/2 - 12\zeta + 6\zeta^2$ | 1 | 2 | 1 | 1 | 1 | $1+i$ | 1 | $2+i$ | 1 | $1+i$ | $2-i$ | $2+i$ |
| | | 2 | $1+2i$ | 1 | $1+i$ | $2+i$ | $2i$ | $2-i$ | $2+i$ | $1+i$ | $i$ | $2+i$ | $1+i$ |
| | $8 - 16\zeta + 8\zeta^2$ | 1 | $2+i$ | 1 | 2 | 1 | $1+i$ | 2 | $2+i$ | 2 | $1+2i$ | $2-i$ | $2+i$ |
| | | $1+i$ | $1+2i$ | 1 | $2+i$ | $2+i$ | $2i$ | 2 | $1+2i$ | | | | |
| | $21/2 - 24\zeta + 14\zeta^2$ | $2+i$ | $1+2i$ | 2 | $2+i$ | 2 | $2+i$ | $2-i$ | 2 | | | | |

This table has been arranged according to increasing values of $\psi(\frac{1}{2})$. For each such value $\psi(\frac{1}{2})$ it was moreover possible to arrange the rows according to increasing values of $(\frac{3}{4} - \zeta)^2 \psi$ for all values of $\zeta$ in $0 \leq \zeta \leq \frac{1}{2}$; e.g. in the first set

$$\tfrac{3}{2} - 2\zeta^2 \leq 2 - 2\zeta \leq \tfrac{3}{2} - 4\zeta + 2\zeta^2 \leq 3 - 6\zeta + 4\zeta^2, \text{ if } 0 \leq \zeta \leq \tfrac{1}{2}.$$

Hence for a given value of $j$ in $2 \leq j \leq 6$, the maximum $\zeta = \zeta_{max}$ belongs to one of those five equations

$$\psi(\zeta_{max}) = j \tag{78}$$

where the function $(\frac{3}{4} - \zeta^2)\,\psi$ is either at the beginning or at the end of one of the three sets of rows of the table [8]).

For given $j$ with $2 \leq j \leq 6$, there is no difficulty in deciding which equation (78) has the root $\zeta_{max}$. The result is contained in the following table, together with the value of $m(j) = (\frac{3}{4} - \zeta^2_{max})^{-\frac{1}{2}}$ and the terms $\alpha$, $\beta$, $\gamma$, $\delta$ of the matrix $\Omega$ belonging to it. (See next page).

In this table, the numbers $\sigma_k$ are defined by

$$\sigma_0 = 2,\ \sigma_1 = 4,\ \sigma_2 = 6,$$

and the numbers $j_n$ by

$$j_1 = \sqrt{6} = 2.44\ldots., \quad j_2 = \frac{49(46 + 45\sqrt{50})}{3582} = 4.98\ldots.$$

In the intervals No. 1—4, the functions $\zeta_{max}$ and $m(j)$ behave in the following manner:

$\zeta_{max}$ and $m(j)$ are both steadily decreasing in the intervals No. 1 and 3;

$\zeta_{max}$ and $m(j)$ are both steadily increasing in the intervals No. 2 and 4.

Further

$$\zeta_{max} = \tfrac{1}{2}, \quad m(j) = \sqrt{2} = 1.41\ldots. \quad \text{for } j = \sigma_0, \sigma_1, \sigma_2,$$

$$\zeta_{max} = 0.27\ldots., \quad m(j) = 1.17\ldots. \quad \text{for } j = j_1,$$

and

$$\zeta_{max} = 0.4\ldots., \quad m(j) = 1.3\ldots. \quad \text{for } j = j_2.$$

[8]) The first poynomial $^3/_2 - 2\xi^2$ of the table can be omitted, because for it $\psi \equiv 2$ identically in $\zeta$.

$m(j)$ for $2 \leq j \leq 6$

| No. | Interval | $(\frac{3}{4} - \zeta^2)\,\psi$ | $\zeta_{max}$ | $m(j) = (\frac{3}{4} - \zeta^2_{max})^{-\frac{1}{2}}$ | Critical pairs given by $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|---|---|---|---|---|---|---|---|---|
| 1 | $\sigma_0 \leq j \leq j_1$ | $3 - 6\zeta + 4\zeta^2$ | $\dfrac{6 - (3j^2 - 12)^{\frac{1}{2}}}{2(j+4)}$ | $\dfrac{j+4}{\{6(j+1) + 3(3j^2 - 12)^{\frac{1}{2}}\}^{\frac{1}{2}}}$ | $1$ | $1$ | $1 - i$ | $1$ |
| | | | | | $1 + i$ | $i$ | $1$ | $1 + i$ |
| 2 | $j_1 \leq j \leq \sigma_1$ | $3/2 + 2\zeta^2$ | $\left\{\dfrac{3(j-2)}{4(j+2)}\right\}^{\frac{1}{2}}$ | $\left\{\dfrac{j+2}{3}\right\}^{\frac{1}{2}}$ | $0$ | $1$ | $1$ | $0$ |
| | | | | | $1$ | $i$ | $0$ | $1$ |
| 3 | $\sigma_1 \leq j \leq j_2$ | $21/2 - 24\zeta + 14\zeta^2$ | $\dfrac{24 - (3j^2 - 12)^{\frac{1}{2}}}{2(j+14)}$ | $\dfrac{j+14}{\{3(2+7j) + 12(3j^2 - 12)^{\frac{1}{2}}\}^{\frac{1}{2}}}$ | $2 + i$ | $1 + 2i$ | $2$ | $2 + i$ |
| | | | | | $2$ | $2 + i$ | $2 - i$ | $2$ |
| 4 | $j_2 \leq j \leq \sigma_2$ | $2 + 2\zeta$ | $\dfrac{(3j^2 - 8j + 4)^{\frac{1}{2}} - 2}{2j}$ | $\{2(j-1) - (3j^2 - 8j + 4)^{\frac{1}{2}}\}^{\frac{1}{2}}$ | $0$ | $i$ | $i$ | $0$ |
| | | | | | $0$ | $i$ | $-1$ | $i$ |
| | | | | | $1$ | $-1 + i$ | $-i$ | $1$ |
| | | | | | $1$ | $-1$ | $0$ | $i$ |

By the table, the graph of the function $m(j)$ is a saw-like curve for $2 \leq j \leq 6$. There would be no difficulty in extending this table to values of $j$ beyond $\sigma_2 = 6$; but this work would be increasingly laborious.

The analogy between our result for two Hermitean forms and that for two quadratic forms [9]) is remarkable.

(Ingekomen 22-8-'46).

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[9]) Cf. K. MAHLER, „Lattice Points in Two-dimensional Star Domains III", l.c [1]).