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On Dyson's improvement of the Thue-Siegel theorem

BY

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K. MAHLER: *On Dyson's improvement of the Thue-Siegel theorem.*

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Two years ago, F. J. DYSON proved the following result [1]):
"*If ξ is an algebraic number of degree $n \geq 2$, if μ is a positive number, and if there are infinitely many rational numbers $\frac{p}{q}$ such that*

$$p, q \text{ are integers}, \quad q \geqslant 1, \quad \left|\xi - \frac{p}{q}\right| < q^{-\mu},$$

then

$$\mu \leqq \sqrt{2n}."$$

This result is stronger than that of C. L. SIEGEL [2]), namely

$$\mu \leqslant \min_{s=1,2,\ldots,n-1} \left(\frac{n}{s+1} + s\right) < 2\sqrt{n}.$$

DYSON obtained his improved inequality by means of a new method for studying the zero points of a polynomial in two variables. As his own proof is given in a somewhat involved form, I present in this paper a simplified proof for his main lemma (Theorem 1). Moreover, since this proof is purely algebraic, I deal always with the case of an arbitrary constant field of characteristic *zero*. This restriction is a natural one, since neither Theorem 1, nor the Thue–Siegel theorem, hold generally for fields of positive characteristic.

P.S. Since the time earlier this year when I wrote the present paper, a new proof of DYSON's result has been published by TH. SCHNEIDER [3]). This proof applies the deaper arithmetical properties of divisibility and may prove more powerful [4]).

[1] In this paper, K denotes a fixed field of characteristic zero; $K[x]$, $K[y]$, and $K[x, y]$, are the rings of all polynomials in x, in y, or in x and y, respectively, with coefficients in K; and $K(x)$ denotes the field of all rational functions in x with coefficients in K. The terms "dependent" and "independent" always mean, "linearly dependent" and "linearly independent" over K.

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1) Acta Mathematica **79**, 225—240 (1947).

2) Mathematische Zeitschrift **10**, 173—213 (1921).

3) Mathematische Nachrichten **2**, 288—295 (1949).

4) Still another proof and a generalization of Dyson's theorem was given by A. O. GELFOND (Vestnik MGU **9**, 3 (1948)), but I have not seen his paper.

[2] We define differentiation in $K(x)$ in the usual formal way. Let $u_0(x), u_1(x), \ldots, u_{l-1}(x)$ be a finite set of elements of $K(x)$; the determinant

$$\left| \frac{d^{\mu} u_{\lambda}(x)}{dx^{\mu}} \right|_{\lambda, \mu = 0, 1, \ldots, l-1}$$

is then called the Wronski determinant of these elements and is denoted by

$$\langle u_0, u_1, \ldots, u_{l-1} \rangle.$$

One easily verifies that if $\varphi(x)$ is any further element of $K(x)$, then

$$\langle \varphi u_0, \varphi u_1, \ldots, \varphi u_{l-1} \rangle = \varphi(x)^l \langle u_0, u_1, \ldots, u_{l-1} \rangle.$$

Lemma 1: *The Wronski deteminant of any finite number of elements of $K(x)$ vanishes identically in x if, and only if, these elements are dependent.*

Proof: If

$$\sum_{\lambda=0}^{l-1} c_{\lambda} u_{\lambda}(x) \equiv 0, \qquad \text{where } c_{\lambda} \in K,$$

then

$$\sum_{\lambda=0}^{l-1} c_{\lambda} \frac{d^{\mu} u_{\lambda}(x)}{dx^{\mu}} \equiv 0 \qquad (\mu = 0, 1, \ldots, l-1),$$

whence $\langle u_0, u_1, \ldots, u_{l-1} \rangle \equiv 0$.

Next assume that $\langle u_0, u_1, \ldots, u_{l-1} \rangle \equiv 0$; we must show that $u_0(x)$, $u_1(x), \ldots, u_{l-1}(x)$ are dependent. This assertion is obvious for $l = 1$; assume it has already been proved for all systems of less than l rational functions. We may exclude the case that $u_0(x) \equiv 0$ since then the Wronski determinant certainly vanishes. Hence

$$u_0(x)^{-l} \langle u_0, u_1, \ldots, u_{l-1} \rangle = \left\langle 1, \frac{u_1}{u_0}, \frac{u_2}{u_0}, \ldots, \frac{u_{l-1}}{u_0} \right\rangle =$$

$$= \left\langle \frac{d(u_1/u_0)}{dx}, \frac{d(u_2/u_0)}{dx}, \ldots, \frac{d(u_{l-1}/u_0)}{dx} \right\rangle \equiv 0.$$

Therefore, by the induction hypothesis, there exist $l-1$ elements $c_1, c_2, \ldots, c_{l-1}$ of K not all zero such that

$$c_1 \frac{d(u_1/u_0)}{dx} + c_2 \frac{d(u_2/u_0)}{dx} + \ldots + c_{l-1} \frac{d(u_{l-1}/u_0)}{dx} \equiv 0.$$

Since the characteristic of K is zero, this implies that

$$c_0 + c_1 \frac{u_1(x)}{u_0(x)} + c_2 \frac{u_2(x)}{u_0(x)} + \ldots + c_{l-1} \frac{u_{l-1}(x)}{u_0(x)} \equiv 0$$

for some element c_0 of K, whence the assertion.

[3] Let now $u_0(x)$, $u_1(x)$, ..., $u_{l-1}(x)$ be a finite set of independent polynomials in $K[x]$, and assume that $u_0(x)$ is of the highest degree amongst these, the degree d_0, say. Then constants $c_1, c_2, \ldots, c_{l-1}$ in K can be found such that

$$u_\lambda^{(1)}(x) = c_\lambda u_0(x) + u_\lambda(x) \qquad (\lambda = 1, 2, \ldots, l-1)$$

are all of degree less than d_0. Assume that $u_1(x)$ is of highest degree, d_1 say, amongst these $l-1$ polynomials. Then constants $c_2^{(1)}, c_3^{(1)}, \ldots, c_{l-1}^{(1)}$ in K can be found such that the $l-2$ polynomials

$$u_\lambda^{(2)}(x) = c_\lambda^{(1)} u_1^{(1)}(x) + u_\lambda^{(1)}(x) \qquad (\lambda = 2, 3, \ldots, l-1)$$

are all of degree less than d_1. Assume that $u_2^{(2)}(x)$ is of highest degree, d_2 say, amongst these polynomials. Then constants $c_3^{(2)}, c_4^{(2)}, \ldots, c_{l-1}^{(2)}$ can be found such that the $l-3$ polynomials

$$u_\lambda^{(3)}(x) = c_\lambda^{(2)} u_2^{(2)}(x) + u_\lambda^{(2)}(x) \qquad (\lambda = 3, 4, \ldots, l-1)$$

are all of degree less than d_2. Continuing in this way, we obtain a set of l polynomials

$$u_0(x), u_1^{(1)}(x), u_2^{(2)}(x), \ldots, u_{l-1}^{(l-1)}(x)$$

of degrees

$$d_0, \ d_1, \ d_2, \ldots, d_{l-1}$$

respectively, where

$$d_0 > d_1 > d_2 > \ldots > d_{l-1}.$$

By the construction, each polynomial $u_\lambda^{(\lambda)}(x)$ differs from $u_\lambda(x)$ only by a linear expression in $u_0(x)$, $u_1(x)$, ..., $u_{\lambda-1}(x)$ with coefficients in K. Hence, by a simple property of determinants, the identity

$$\langle u_0, u_1, \ldots, u_{l-1} \rangle = \langle u_0, u_1^{(1)}, \ldots, u_{l-1}^{(l-1)} \rangle$$

holds.

Lemma 2: *Let $u_0(x)$, $u_1(x)$, ..., $u_{l-1}(x)$ be polynomials in $K[x]$ of degrees not greater than d. Then the Wronski determinant*

$$\langle u_0, u_1, \ldots, u_{l-1} \rangle$$

is a polynomial of degree not greater than $l(d-l+1)$.

Proof: It suffices to prove the assertion when the polynomials are independent. The polynomials

$$u_0(x), u_1^{(1)}(x), \ldots, u_{l-1}^{(l-1)}(x),$$

as just constructed, have degrees

$$d_0 \leqslant d-0, \ d_1 \leqslant d_1 - 1, \ldots, d_{l-1} \leqslant d-(l-1).$$

Furthermore, the Wronski determinant $\langle u_0, u_1^{(1)}, \ldots, u_{l-1}^{(l-1)} \rangle$ is a sum of $l!$ terms of the form

$$\mp \frac{d^{i_0} u_0(x)}{dx^{i_0}} \frac{d^{i_1} u_1^{(1)}(x)}{dx^{i_1}} \cdots \frac{d^{i_{l-1}} u_{l-1}^{(l-1)}(x)}{dx^{i_{l-1}}}$$

where $i_0, i_1, \ldots, i_{l-1}$ run over all permutations of $0, 1, \ldots, l-1$. Each such term is of degree

$$\sum_{\lambda=0}^{l-1} (d_\lambda - i_\lambda) = \sum_{\lambda=0}^{l-1} \{d_\lambda - (l-\lambda-1)\} \leq \sum_{\lambda=0}^{l-1} \{(d-\lambda) - (l-\lambda-1)\} = l(d-l+1),$$

whence the assertion.

[4] If $P(x, y)$ is any polynomial in $K[x, y]$, then write

$$P_{ij}(x, y) = \frac{\partial^{i+j} P(x, y)}{i!\, j!\, \partial x^i\, \partial y^j} \qquad (i, j = 0, 1, 2, \ldots).$$

We denote by r and s two positive integers which will be fixed in the next section, by ξ and η two elements of K, and by ϑ a non-negative real number. We then say that $P(x, y)$ is at least of index ϑ at (ξ, η) if

$$P_{ij}(\xi, \eta) = 0 \text{ for } i \geq 0, j \geq 0, \frac{i}{r} + \frac{j}{s} < \vartheta;$$

in the special case $\vartheta = 0$, there are no conditions.

This definition can be replaced by an equivalent one, as follows. Denote by z an indeterminate. Then

$$P(\xi + xz^s, \eta + yz^r) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{ij}(\xi, \eta)\, x^i y^j z^{rs\left(\frac{i}{r} + \frac{j}{s}\right)}, = P\langle z \rangle \text{ say},$$

becomes a polynomial in z with coefficients in $K[x, y]$. This formula shows that $P(x, y)$ is at least of index ϑ at (ξ, η) if, and only if, $P\langle z \rangle$ is divisible by $z^{rs\vartheta}$ (i.e. all powers of z occurring in $P\langle z \rangle$ must have exponents not less than $rs\vartheta$). If we multiply several such expressions

$$P_0\langle z \rangle, P_1\langle z \rangle, \ldots, P_{l-1}\langle z \rangle$$

which are divisible by

$$z^{rs\vartheta_0}, z^{rs\vartheta_1}, \ldots, z^{rs\vartheta_{l-1}},$$

respectively, then the product is divisible by

$$z^{rs(\vartheta_0 + \vartheta_1 + \ldots + \vartheta_{l-1})}.$$

Therefore the following result holds:

Lemma 3: *If, for $\lambda = 0, 1, \ldots, l-1$, the polynomial $P_\lambda(x, y)$ in $K[x, y]$ is at least of index ϑ_λ at (ξ, η), then*

$$P_0(x, y)\, P_1(x, y) \ldots P_{l-1}(x, y)$$

is at least of index

$$\vartheta_0 + \vartheta_1 + \ldots + \vartheta_{l-1}$$

at (ξ, η).

[5] From now on,

$$R(x, y) = \sum_{h=0}^{r} \sum_{k=0}^{s} R_{hk} x^h y^k \not\equiv 0$$

is a fixed polynomial in $K[x, y]$ of degrees not greater than r in x and s in y; here r and s are given positive integers. We further denote by

$$\theta_0, \theta_1, \ldots, \theta_n \qquad (n \geqslant 0)$$

a finite number of real numbers satisfying

$$0 < \theta_f \leqslant 1 \qquad (f = 0, 1, \ldots, n),$$

and by

$$\xi_0, \xi_1, \ldots, \xi_n \quad \text{and} \quad \eta_0, \eta_1, \ldots, \eta_n$$

two sets, each of $n+1$ elements of K, such that no two elements of the same set are equal.

Throughout this note, we make the assumption that $R(x, y)$ is, for $f = 0, 1, \ldots, n$, at least of index θ_f at (ξ_f, η_f), so that

$$R_{ij}(\xi_f, \eta_f) = 0 \quad \text{if} \quad i \geqslant 0, j \geqslant 0, \frac{i}{r} + \frac{j}{s} < \theta_f, f = 0, 1, \ldots, n.$$

[6] Since

$$R(x, y) = \sum_{k=0}^{s} \left(\sum_{h=0}^{r} R_{hk} x^h \right) y^k,$$

the polynomial can be written in the form

$$R(x, y) = \sum_{\lambda=0}^{l-1} u_\lambda(x) v_\lambda(y),$$

where the u's are elements of $K[x]$ of degrees not greater than r, the v's are polynomials in $K[y]$ of degrees not greater than s, and where

$$1 \leqslant l \leqslant \min(r, s) + 1.$$

Amongst all representations of this form, select one for which the number l of terms is a minimum. Then both the l polynomials

$$u_0(x), \ u_1(x), \ldots, u_{l-1}(x),$$

and the l polynomials

$$v_0(y), \ v_1(y), \ldots, v_{l-1}(y),$$

are independent. For if, say, the u's are not independent, then we may assume that $u_{l-1}(x)$ can be written as

$$u_{l-1}(x) = \sum_{\lambda=0}^{l-2} \alpha_\lambda u_\lambda(x)$$

where the coefficients α_λ lie in K; therefore

$$R(x, y) = \sum_{\lambda=0}^{l-2} u_\lambda(x) \{v_\lambda(y) + \alpha_\lambda v_{l-1}(y)\}$$

becomes a sum of only $l-1$ terms, contrary to the definition of l.

We conclude therefore from Lemma 1 that neither of the two Wronski determinants

$$U(x) = \langle u_0(x), u_1(x), \ldots, u_{l-1}(x) \rangle \text{ and } V(y) = \langle v_0(y), v_1(y), \ldots, v_{l-1}(y) \rangle$$

vanishes identically. Moreover, by Lemma 2,

$$U(x) \text{ is at most of degree } l(r-l+1) \text{ in } x,$$

and

$$V(y) \text{ is at most of degree } l(s-l+1) \text{ in } y.$$

[7] Denote by $(x-\xi_f)^{r_f}$, where $f = 0, 1, \ldots, n$, the highest power of $x - \xi_f$ dividing $U(x)$, and by $(y - \eta_f)^{s_f}$, where $f = 0, 1, \ldots, n$, the highest power of $y - \eta_f$ dividing $V(y)$. Since all the ξ's and also all the η's are different, $U(x)$ is divisible by

$$\prod_{f=0}^{n} (x - \xi_f)^{r_f},$$

and $V(y)$ is divisible by

$$\prod_{f=0}^{n} (y - \eta_f)^{s_f}.$$

Therefore, on comparing the degrees, we obtain the two inequalities,

$$\left.\begin{aligned} r_0 + r_1 + \ldots + r_n &\leqslant l(r-l+1), \\ s_0 + s_1 + \ldots + s_n &\leqslant l(s-l+1). \end{aligned}\right\} \quad \text{(I)}$$

[8] We next introduce the determinant

$$W(x, y) = |R_{\varkappa\mu}(x, y)|_{\varkappa, \mu = 0, 1, \ldots, l-1}.$$

Since

$$R_{\varkappa\mu}(x, y) = \frac{1}{\varkappa!\,\mu!} \sum_{\lambda=0}^{l-1} u_\lambda^{(\varkappa)}(x)\, v_\lambda^{(\mu)}(y),$$

the product rule of determinants leads to the identity,

$$U(x)\, V(y) = \{1!\, 2! \ldots (l-1)!\}^2\, W(x, y),$$

so that also $W(x, y)$ does not vanish identically.

[9] Let f be one of the indices 0, 1, ..., n. Then, by hypothesis, $R(x, y)$ is at least of index θ_f at (ξ_f, η_f); therefore $R_{\varkappa\mu}(x, y)$ is at least of index

$$\max\left(0, \theta_f - \frac{\varkappa}{r} - \frac{\mu}{s}\right)$$

at (ξ_f, η_f).

Now $W(x, y)$ is a sum of $l!$ terms of the form

$$\mp R_{i_0 0}(x, y)\, R_{i_1 1}(x, y) \ldots R_{i_{l-1} l-1}(x, y),$$

where $i_0, i_1, \ldots, i_{l-1}$ run over all permutations of $0, 1, \ldots, l-1$. By Lemma 3, such a term is at least of index

$$\sum_{\lambda=0}^{l-1} \max\left(0, \theta_f - \frac{i_\lambda}{r} - \frac{\lambda}{s}\right) \geqslant \sum_{\lambda=0}^{l-1} \max\left(-\frac{i_\lambda}{r}, \theta_f - \frac{i_\lambda}{r} - \frac{\lambda}{s}\right) =$$
$$= \sum_{\lambda=0}^{l-1} \max\left(0, \theta_f - \frac{\lambda}{s}\right) - \sum_{\lambda=0}^{l-1} \frac{i_\lambda}{r}$$

at (ξ_f, η_f). Since

$$\sum_{\lambda=0}^{l-1} \frac{i_\lambda}{r} = \sum_{\lambda=0}^{l-1} \frac{\lambda}{r} = \frac{l(l-1)}{2r},$$

the whole determinant $W(x, y)$ is therefore also at least of index

$$\sum_{\lambda=0}^{l-1} \max\left(0, \theta_f - \frac{\lambda}{s}\right) - \frac{l(l-1)}{2r}$$

at (ξ_f, η_f).

[10] On the other hand, $U(x)V(y)$ is divisible exactly by

$$(x-\xi_f)^{r_f}(y-\eta_f)^{s_f},$$

so that

$$\left.\frac{\partial^{i+j}\{U(x)V(y)\}}{i!\,j!\,\partial x^i\,\partial y^j}\right|_{x=\xi_f,\, y=\eta_f} \begin{cases} = 0 \text{ if } i \geqslant 0, j \geqslant 0, \dfrac{i}{r} + \dfrac{j}{s} < \dfrac{r_f}{r} + \dfrac{s_f}{s}, \\ \neq 0 \text{ if } i = r_f, j = s_f. \end{cases}$$

From the identity

$$U(x)\,V(y) = \{1!\,2!\ldots(l-1)!\}^2\, W(x, y),$$

we therefore deduce the relations

$$\sum_{\lambda=0}^{l-1} \max\left(0, \theta_f - \frac{\lambda}{s}\right) - \frac{l(l-1)}{2r} \leqslant \frac{r_f}{r} + \frac{s_f}{s} \qquad (f = 0, 1, \ldots, n).$$

On adding these $n+1$ inequalities and the two inequalities (I), we obtain the final inequality

$$\sum_{f=0}^{n} \sum_{\lambda=0}^{l-1} \max\left(0, \theta_f - \frac{\lambda}{s}\right) \leqslant (n+1)\frac{l(l-1)}{2r} + \frac{l(r-l+1)}{r} + \frac{l(s-l+1)}{s}, \quad \text{(II)}$$

where now the unknown degrees r_f and s_f no longer occur.

[11] The double sum on the left-hand side of (II) is easily replaced by a simple one. Put

$$\Lambda_f = \min([\theta_f s]+1, l) \qquad (f=0, 1, \ldots, n),$$

so that

$$\max\left(0, \theta_f - \frac{\lambda}{s}\right) = \begin{cases} \theta_f - \frac{\lambda}{s} & \textit{if} \quad 0 \leqslant \lambda \leqslant \Lambda_f - 1, \\ 0 & \textit{if} \quad \lambda \geqslant \Lambda_f. \end{cases}$$

Therefore

$$\sum_{\lambda=0}^{l-1} \max\left(0, \theta_f - \frac{\lambda}{s}\right) = \sum_{\lambda=0}^{\Lambda_f - 1}\left(\theta_f - \frac{\lambda}{s}\right) = \tfrac{1}{2}\Lambda_f\left(2\theta_f - \frac{\Lambda_f - 1}{s}\right),$$

so that the left-hand side of (II) may be written as

$$\tfrac{1}{2}\sum_{f=0}^{n} \Lambda_f\left(2\theta_f - \frac{\Lambda_f - 1}{s}\right).$$

In order to simplify further, put

$$X = \frac{l}{s}, \qquad X_f = \min(\theta_f, X) \qquad (f=0, 1, \ldots, n).$$

Then

$$s X_f = \min(s\theta_f, sX) = \min(s\theta_f, l)$$

and

$$\Lambda_f - 1 \leqslant s X_f \leqslant \Lambda_f, \text{ hence } \Lambda_f\left(2\theta_f - \frac{\Lambda_f - 1}{s}\right) \geqslant s X_f (2\theta_f - X_f).$$

Therefore (II) implies that

$$\frac{s}{2}\sum_{f=0}^{n} X_f(2\theta_f - X_f) \leqslant (n+1)\frac{l(l-1)}{2r} + \frac{l(r-l+1)}{r} + \frac{l(s-l+1)}{s}.$$

Next, the right-hand side of this inequality may be written as

$$(n+1)\frac{l(l-1)}{2r} + \frac{l(r-l+1)}{r} + \frac{l(s-l+1)}{s} = \left(2l - \frac{l^2}{s}\right) + \\ + \left(\frac{l}{s} + (n-1)\frac{l(l-1)}{2r}\right) = s(2X - X^2)\left\{1 + \frac{1}{2-X}\left(\frac{1}{s} + \frac{(n-1)(l-1)}{2r}\right)\right\}.$$

Because, by [6],

$$l \leqq \min(r, s) + 1 \leqq s + 1,$$

the inequality becomes therefore

$$\sum_{f=0}^{n} X_f (2\theta_f - X_f) \leqq 2\{1-(1-X)^2\}\left\{1 + \frac{1}{2-X}\left(\frac{1}{s} + \frac{(n-1)s}{2r}\right)\right\}.$$

[12] So far, r and s have been left arbitrary. Let now δ be a number satisfying

$$0 < \delta \leqq 1,$$

and restrict r and s by the conditions,

$$s \geqq \frac{5}{\delta} \geqq 5, \qquad r \geqq \frac{5(n-1)s}{2\delta}.$$

Then

$$X = \frac{l}{s} \leqq \frac{s+1}{s} \leqq 1 + \tfrac{1}{5}, \quad 2 - X \geqq \tfrac{4}{5}, \quad \frac{1}{s} \leqq \frac{\delta}{5}, \quad \frac{(n-1)s}{2r} \leqq \frac{\delta}{5},$$

so that

$$\frac{1}{2-X}\left(\frac{1}{s} + \frac{(n-1)s}{2r}\right) \leqq \frac{5}{4}\left(\frac{\delta}{5} + \frac{\delta}{5}\right) = \frac{\delta}{2},$$

and our inequality takes the simple form

$$\sum_{f=0}^{n} X_f (2\theta_f - X_f) \leqq (2+\delta)\{1-(1-X)^2\}.$$

But, for $f = 0, 1, \ldots, n$,

$$X_f (2\theta_f - X_f) - \theta_f^2\{1-(1-X)^2\} \equiv \theta_f^2 (1-X)^2 - (\theta_f - X_f)^2$$

is not negative, since either $X \geqq \theta_f$, when $X_f = \theta_f$ and

$$\theta_f^2 (1-X)^2 - (\theta_f - X_f)^2 = \theta_f^2 (1-X)^2 \geqq 0;$$

or $X < \theta_f$, when $X_f = X$ and $X \leqq 1$ and therefore

$$\theta_f^2 (1-X)^2 - (\theta_f - X_f)^2 \equiv X(1-\theta_f)\{\theta_f(1-X) + (\theta_f - X)\} \geqq 0.$$

Hence

$$\{1-(1-X)^2\} \sum_{f=0}^{n} \theta_f^2 \leqq \sum_{f=0}^{n} X_f (2\theta_f - X_f) \leqq (2+\delta)\{1-(1-X)^2\},$$

and since $(1-X)^2 < 1$, we obtain finally the result,

$$\sum_{f=0}^{n} \theta_f^2 \leqq 2 + \delta.$$

Our discussion has thus led us to the following theorem:

Theorem 1: *Let $\delta, \theta_0, \theta_1, \ldots, \theta_n$ be $n+2$ real numbers satisfying*

$$0 < \delta \leqslant 1,\ 0 < \theta_0 \leqslant 1,\ 0 < \theta_1 \leqslant 1, \ldots, 0 < \theta_n \leqslant 1,$$

and let r and s be two integers satisfying

$$s \geqslant \frac{5}{\delta}, \qquad r \geqslant \frac{5\,(n-1)\,s}{2\,\delta}.$$

Let

$$R\,(x, y) \not\equiv 0$$

be a polynomial of degrees not greater than r in x and s in y, with coefficients in a field K of characteristic zero; write

$$R_{ij}\,(x, y) = \frac{\partial^{i+j}\,R\,(x, y)}{i!\,j!\,\partial x^i\,\partial y^j} \qquad (i, j = 0, 1, 2, \ldots).$$

Further let

$$\xi_0, \xi_1, \ldots, \xi_n \quad \textit{and} \quad \eta_0, \eta_1, \ldots, \eta_n$$

be two sets, each of $n+1$ elements of K, such that no two elements of the same set are equal. If now

$$R_{ij}\,(\xi_f, \eta_f) = 0 \text{ for } i \geqslant 0, \quad j \geqslant 0, \quad \frac{i}{r} + \frac{j}{s} < \theta_f, \quad f = 0, 1, \ldots, n,$$

then

$$\theta_0^2 + \theta_1^2 + \ldots + \theta_n^2 \leqslant 2 + \delta.$$

In a second paper, I shall prove an analoguous theorem for polynomials of the form

$$\Sigma\,\Sigma\,R_{hk}\,x^h\,y^k \qquad \left(h \geqslant 0, k \geqslant 0, \frac{h}{r} + \frac{k}{s} \leqslant 1\right),$$

and apply this result to the study of the continued fractions of algebraic numbers.

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