

FAREY SECTIONS IN THE FIELDS OF GAUSS AND EISENSTEIN

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In the study of the approximation of irrational numbers by rational ones, one is led to consider certain special sets of rational numbers, the so-called Farey sections. If N is any positive integer, then the N th Farey section \mathfrak{F}_N consists of all *different* fractions $x = a/b$ for which

$$|a| \leq N, \quad |b| \leq N.$$

We allow a or b to vanish, but exclude the case that both are zero. In particular, \mathfrak{F}_N contains the improper element

$$\frac{a}{0} = \infty \quad (a \neq 0),$$

where the sign of a is immaterial.

With \mathfrak{F}_N , we associate a subdivision of the infinite real axis into a finite number of subsets, as follows.

If a/b is any element of \mathfrak{F}_N , where without loss of generality $(a, b) = 1$, then the generalized distance $|x; a/b|$ of an arbitrary real point x from a/b is defined by

$$\left| x; \frac{a}{b} \right| = |bx - a| = |b| \left| x - \frac{a}{b} \right|.$$

Next $\mathfrak{R}(a/b)$ is the set of all real numbers x for which $|x; a/b| \leq |x; a'/b'|$ for all $a'/b' \in \mathfrak{F}_N$, thus which are nearest to a/b with respect to its distance function. There are thus as many sets $\mathfrak{R}(a/b)$ as there are different elements a/b of \mathfrak{F}_N .

We can arrange the elements of \mathfrak{F}_N according to increasing size and may then speak of consecutive elements of \mathfrak{F}_N .

The following theorems are all well known:

A. If $a'/b', a/b, a''/b''$, where $b' \geq 0, b > 0, b'' \geq 0$, are consecutive elements of \mathfrak{F}_N , then $\mathfrak{R}(a/b)$ is the interval

$$\frac{a + a'}{b + b'} \leq x \leq \frac{a + a''}{b + b''};$$

on the other hand, $\mathfrak{R}(1/0)$ consists of the points satisfying

$$x \leq -(N + 1) \text{ or } x \geq (N + 1).$$

B. If $b > 0, b' > 0$, then a/b and a'/b' are consecutive elements of \mathfrak{F}_N if and only if

$$(1) \quad ab' - a'b = \mp 1;$$

(2) *the median $(a + a')/(b + b')$ is not in \mathfrak{S}_N .*

C. *All terms of \mathfrak{S}_{N+1} are either in \mathfrak{S}_N or are medians of consecutive terms of \mathfrak{S}_N .*

D. *If a/b is in \mathfrak{S}_N , and $|a/b| \leq 1$, then $|z; a/b| \leq 1/N$ for $z \in \mathfrak{R}(a/b)$.*

From D, one easily deduces Minkowski's theorem on two linear forms.

I shall not speak here on this classical theory, but give you instead some information about a very similar theory in the fields $k(i)$ of Gauss and $k(\rho)$ of Eisenstein. The results in Gauss's field are rather more difficult, so that I shall go more into detail in this case.

First I make some remarks on the history of the problem. In 1940 I obtained empirically all the results which I am going to discuss, and I also had a good guess as to how to obtain proofs. But other work kept me from occupation with this question, and I took it up again only in 1948 and 1949, being greatly helped by two of my Manchester colleagues, W. Ledermann and I. W. S. Cassels. These two finally obtained complete proofs for all my guesses, and it was Cassels who put our paper into its final form.¹

Let, say, $k(i)$ be the Gaussian field, and let N be a positive integer; we might assume, for simplicity, that N is the norm of an integral element of $k(i)$, but shall not do so. Denote by \mathfrak{S}_N the set of all simplified different fractions a/b where a, b are integers in $k(i)$ not both zero of norm not greater than N . We include in \mathfrak{S}_N the improper point $1/0 = \infty$ and call \mathfrak{S}_N the Farey section of order N ; it consists thus of a finite number of points in the complex plane, including the point at infinity.

Let now a/b be any element of \mathfrak{S}_N , so that $(a, b) = 1$, that is, a and b are relatively prime. We associate with the point a/b a distance function

$$|z; a/b| = |bz - a| = |b| \cdot |z - a/b|,$$

giving the distance of an arbitrary complex point z from a/b ; in particular

$$|z; 1/0| = 1 \text{ for all } z.$$

We next define $\mathfrak{R}(a, b)$ as the set of all complex points z for which

$$\left| z; \frac{a}{b} \right| \leq \left| z; \frac{a'}{b'} \right| \quad \text{for all } \frac{a'}{b'} \in \mathfrak{S}_N.$$

Then \mathfrak{S}_N and the corresponding system of sets $\mathfrak{R}(a/b)$, where $a/b \in \mathfrak{S}_N$, are invariant under the mappings

$$z \rightarrow i^k z, \quad z \rightarrow i^k / z, \quad z \rightarrow \bar{z}$$

of the complex plane.

The points in the complex plane are not ordered; therefore in order to study the sets $\mathfrak{R}(a/b)$ we must use different methods than in the real case, essentially

¹ Our joint paper will soon appear in the Transactions of the Royal Society.

a combination of elementary topology with simple arithmetical properties of the Gaussian field.

To this end we must study the boundaries and inner points of the sets $\mathfrak{R}(a/b)$ in detail. It is clear from the definition that no two such sets have inner points in common; they do, however, have possibly common boundary points, and apart from these cover the whole plane without overlapping.

In every boundary point z of $\mathfrak{R}(a/b)$, we must evidently be equidistant from a/b and a second point a'/b' in \mathfrak{S}_N :

$$\left| z; \frac{a}{b} \right| = \left| z; \frac{a'}{b'} \right|,$$

that is,

$$|bz - a| = |b'z - a'|.$$

Hence the boundary of $\mathfrak{R}(a/b)$ consists of arcs of circles or lines, and evidently of only a finite number of these.

A rather lengthy study of $\mathfrak{R}(a/b)$ leads now to the following important result:

A. $\mathfrak{R}(a, b)$ is, in the closed complex plane, a simply connected region, and is even a star domain with respect to the point a/b if $N \geq N_0$.

(This is probably true for all N , as the figures suggest and as we have proved for all regions $\mathfrak{R}(a/b)$ inside the unit circle.)

We call a/b the centre of $\mathfrak{R}(a/b)$. We further say that two regions $\mathfrak{R}(a/b)$ and $\mathfrak{R}(a'/b')$, and their centres a/b and a'/b' , are adjacent if $\mathfrak{R}(a/b)$ and $\mathfrak{R}(a'/b')$ have boundary points (but of course no inner points) in common. The following necessary and sufficient condition holds then:

B. The reduced points a/b and a'/b' in \mathfrak{S}_N are adjacent if and only if

(1) $ab' - a'b = i^k$ or $= i^k(1 + i)$;

(2) not all four medians $(a + i^k a')/(b + i^k b')$ are in \mathfrak{S}_N .

In addition, the following result holds:

C. Every point of \mathfrak{S}_{N+1} is either an element of \mathfrak{S}_N or can be written as the median of two adjacent elements of \mathfrak{S}_N .

We next obtain the following analogue to the real case:

D. If a/b lies in the interior of the unit circle and if $z \in \mathfrak{R}(a/b)$, then

$$\left| z; \frac{a}{b} \right| = |bz - a| \leq \frac{\kappa}{N}$$

where

$$\kappa = \frac{2^{1/2}}{3 - 3^{1/2}} \quad (\kappa > 1),$$

and this is best possible.

From this theorem, we obtain a very short and simple proof of the following well-known result of Minkowski (also proved by E. Hlawka):

E. Let $\alpha, \beta, \gamma, \delta$ be four complex numbers of determinant $\alpha\delta - \beta\gamma = 1$. Then there exist two Gaussian integers x, y not both zero satisfying

$$|\alpha x + \beta y| \leq \kappa,$$

$$|\gamma x + \delta y| \leq \kappa,$$

and here $\kappa = 2^{1/2}/(3 - 3^{1/2})$ is the best possible constant.

It was the search for a simpler proof of this theorem which led me originally to a study of the Farey sections in $k(i)$.

We continue now with the study of the regions $\mathfrak{R}(a/b)$. As already mentioned, their boundary is formed by arcs of circles or lines. At every boundary point of $\mathfrak{R}(a/b)$ the set touches at least one similar set $\mathfrak{R}(a/b)$. We call now a *node* any point in the complex plane where at least three sets $\mathfrak{R}(a/b)$ meet. The result as follows can then be proved:

F. At a node, either three or four regions $\mathfrak{R}(a/b)$ meet. If four regions meet, they subtend equal angles $\pi/2$. If only three regions meet, then they either subtend angles $\pi/2, 3\pi/4, 3\pi/4$, or they subtend angles $2\pi/3, 2\pi/3, 2\pi/3$. In the first two cases, the node is an element of $k(i)$ and, in fact, a median of points in \mathfrak{S}_N ; in the last case, the node does not lie in $k(i)$, but in the biquadratic field $k(i, (-3)^{1/2})$.

The general region $\mathfrak{R}(a/b)$ is found to be a polygon bounded by a finite number of arcs of circles or lines. The number of these sides can be arbitrarily large, depending on N .

What we have found for $k(i)$ applies with very little change also for Farey sections in Eisenstein's field $k(\rho)$ where

$$\rho = \frac{-1 + (-3)^{1/2}}{2},$$

but the existence of six units $\mp\rho^k$ leads to some simplification. Theorem A is unchanged. In Theorem B, the conditions are now

$$(1) \quad ab' - a'b = \mp\rho^k \text{ or } = \mp\rho^k(1 + \rho);$$

$$(2) \quad \text{not all medians } (a + \epsilon\rho^k a')/(b + \epsilon\rho^k b'), \epsilon = \mp 1, \text{ are in } \mathfrak{S}_N.$$

Theorem C is unchanged. In Theorems D and E, κ may be taken equal to 1. In Theorem F, all nodes lie in $k(\rho)$; either four or three regions come together, and the angles are in the first case

$$\frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3},$$

while in the second case they are

$$\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3} \quad \text{or} \quad \frac{\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{6}.$$

The methods used by us are simple and general and there seems little doubt that analogous, although more complicated, results hold for all imaginary quadratic fields. I also think that a similar theory can be developed for the quaternion ring.

Another possible generalization deals with the simultaneous approximations of two real numbers x, y by means of fractions $a/c, b/c$ of the same denominator. We may take, say,

$$\left| x, y; \frac{a}{c}, \frac{b}{c} \right| = c \max \left(\left| x - \frac{a}{c} \right|, \left| y - \frac{b}{c} \right| \right)$$

as the distance of (x, y) from $(a/c, b/c)$. But the theory becomes then very difficult, and the regions $\mathfrak{R}(a/c, b/c)$ are possibly not simply connected. A much simpler theory arises if

$$\left| x, y; \frac{a}{c}, \frac{b}{c} \right| = c \left(\left(x - \frac{a}{c} \right)^2 + \left(y - \frac{b}{c} \right)^2 \right)^{1/2}.$$

I conclude my lecture. You will agree that the generalization of Farey sections to complex fields leads to pretty figures and also to results of interest both in geometry and in the theory of numbers. Moreover there seems little doubt that there is scope for much further work, even in connection with Hermite's troublesome problem of the famous approximations of several real numbers.

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