ON THE GENERATING FUNCTION OF THE INTEGERS WITH A MISSING DIGIT

By K. Mahler

Let n be a positive integer such that no digit in its decimal representation is equal to zero, and let N be the set of all such integers n. It is well known that the series

$$\sigma = \sum_{n \varepsilon N} 1/n$$

converges. Whether its value σ is a transcendental number, or whether it can be expressed by means of elementary transcendental functions, is, however, a difficult question. In this note, I shall dicuss the related series

$$f(z) = \sum_{n \varepsilon N} z^n$$

with which σ is connected by the relation

$$\sigma = \int_0^1 \frac{f(z)}{z} dz.$$

I shall prove that if z is an algebraic number such that

$$0 < |z| < 1,$$

then $f(z)$ is a transcendental number; and a similar result holds for infinitely many similar functions.

1. The problem. Let $q \geqslant 2$ be a fixed positive integer. Every non-negative integer n can be written in a unique way as a q-adic sum

$$n = h_0 + h_1 q + \ldots + h_r q^r = (h_0, h, \ldots, h_r),$$

where $h_0, h_1 \ldots, h_r$ are integers $0, 1, \ldots, q-1$, and where, in particular, $h_r \neq 0$. For $n = 0$, we write $0 = (0)$. Let

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k be a fixed one of the integers $0, 1, \ldots, q-1$, and let $\mathcal{N}(k)$ be the set of all those integers $n \geqslant 0$ whose digits h_ρ are all different from k,

$n = (h_0, h_1, \ldots, h_r) \geqslant 0,\ 0 \leqslant h_\rho \leqslant q-1,\ h_\rho \neq k\ (\rho = 0, 1, \ldots, r).$

We shall study here the properties of the generating function

$$f_k(z) = \sum_{n \varepsilon N(k)} z^n$$

of $\mathcal{N}(k)$.

2. **The functional equation for** $f_k(z)$. It is clear that $f_k(z)$ is majorized by the series $1+z+z^2+\ldots = (1-z)^{-1}$ and so converges absolutely for $|z| < 1$.

There exists a functional equation between $f_k(z)$ and $f_k(z^q)$ which takes different forms for $k=0$ and for $k \neq 0$.

I. $k=0$. If $n=(h_0, h_1, \ldots, h_r)$ belongs to $\mathcal{N}(0)$, then the following two cases arise:

(*i*) $r=0$, $n=h_0$, so that n is one of the integers $1, 2, \ldots, q-1$.

(*ii*) $r \geqslant 1$, so that n can be written as $n = h_0 + qn'$ where $1 \leqslant h_0 \leqslant q-1$, $n' = (h_1, h_2, \ldots, h_r) \in \mathcal{N}(0)$. Therefore

$$f_0(z) = \sum_{h_0=1}^{q-1} \left\{ z^{h_0} + \sum_{n' \varepsilon N(0)} z^{h_0 + qn'} \right\},$$

so that

$$f_0(z) = \frac{z-z^q}{1-z}\left(1+f_0(z^q)\right). \tag{I}$$

II. $k = 1, 2, \ldots, q-1$. If n belongs to $\mathcal{N}(k)$, then we can write

$$n = (h_0, h_1, \ldots, h_r) = h_0 + qn'$$

where h_0 is one of the integers $0, 1, 2, \ldots, k-1, k+1, \ldots, q-1$, and where

$$n' = (h_1, h_2, \ldots h_r) \in \mathcal{N}(k).$$

It is now clear that

$$f_k(z) = \sum_{\substack{h_0=0\\ h_0\neq k}}^{q-1} \sum_{n'\varepsilon N(k)} z^{h_0+qn'},$$

whence

$$f_k(z) = \left(\frac{1-z^q}{1-z} - z^k\right) f_k(z^q). \qquad \text{(II)}$$

The functional equations (I) and (II) may be combined into the one equation

$$f_k(z) = \left(\frac{1-z^q}{1-z} - z^k\right)(\varepsilon_k + f_k(z^q)) \quad (k = 0, 1, \ldots, q-1), \quad (1)$$

where $\varepsilon_k = 1$ if $k = 0$, and $\varepsilon_k = 0$ if $k = 1, 2, \ldots, q-1$. In the simplest case $q = 2$, we have

$$f_0(z) = \sum_{\nu=1}^{\infty} z^{2^\nu - 1}, \qquad f_0(z) = z + zf_0(z^2),$$

$$f_1(z) = 1 \qquad , \qquad f_1(z) = f_1(z^2).$$

3. **The analytic behaviour of $f_k(z)$.** It is clear from the definition that

$$f_0(z) = z + z^2 + \ldots + z^{q-1} + \ldots,$$

$$f_k(z) = 1 + z + \ldots + z^{k-1} + z^{k+1} + \ldots \quad (k = 1, \ldots, q-1),$$

whence, for $|z| < 1$,

$$\lim_{\nu\to\infty} f_k(z^{q^\nu}) = 1 - \varepsilon_k \qquad (k = 0, 1, \ldots, q-1). \qquad (2)$$

We further deduce from the functional equations (I) and (II) that

$$f_0(z) = \frac{z-z^q}{1-z} + \frac{z-z^q}{1-z}\frac{z^q-z^{q^2}}{1-z^q} + \ldots$$
$$+ \frac{z-z^q}{1-z}\frac{z^q-z^{q^2}}{1-z^q}\cdots\frac{z^{q^{\nu-1}}-z^{q^\nu}}{1-z^{q^{\nu-1}}}(1+f_0(z^{q^\nu})), \qquad (3)$$

and

$$f_k(z) = \left(\frac{1-z^q}{1-z} - z^k\right)\left(\frac{1-z^{q^2}}{1-z^q} - z^{kq}\right)\cdots$$
$$\times \left(\frac{1-z^{q^\nu}}{1-z^{q^{\nu-1}}} - z^{kq^{\nu-1}}\right) f_k(z^{q^\nu}), \quad (k = 1, 2, \ldots, q-1). \qquad (4)$$

THEOREM 1. *If the special case $q=2$, $k=1$ is excluded, then $f_k(z)$ is regular inside the unit circle and has this circle as its natural boundary.*

PROOF. Let κ and λ be two non-negative integers; put

$$\theta = e^{\frac{2\pi i\kappa}{q^\lambda}}.$$

Assume that κ is prime to q so that θ is a primitive q^λ-th root of unity. It is obvious that for $\lambda \geqslant 1$ none of the polynomials

$$\frac{z^{q^{\nu-1}}-z^{q^\nu}}{1-z^{q^{\nu-1}}},\quad \frac{1-z^{q^\nu}}{1-z^{q^{\nu-1}}}-z^{kq^{\nu-1}} \qquad (\nu = 1, 2, \ldots, \lambda)$$

in z vanishes if $z=\theta$. On the other hand, if the case $q=2$, $k=1$ is excluded, then evidently

$$\lim_{r\to 1} f_k(r) = +\infty \tag{5}$$

as r tends to 1 along the real interval $0 \leqslant r < 1$. But then, by $\theta^{q^\lambda}=1$, from (3), (4), and (5), also

$$\lim_{r\to 1} f_k(\theta r) = \infty.$$

Now the points θ are everywhere dense on the unit circle, and the assertion follows at once.

COROLLARY. *Except for the case $q=2$, $k=1$, $f_k(z)$ is a transcendental function of z.*

4. The arithmetic behaviour of $f_k(z)$. Some twenty years ago, I proved a result in which the following theorem is contained as a special case [*Mathematische Annalen*, 101 (1929), 332-366].

THEOREM 2. *Let $q \geqslant 2$ be a fixed integer, and let*

$$F(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$$

be a power series with the following properties:

(i) *All a_ν are rational numbers.*

(ii) *$F(z)$ converges in a neighbourhood of $z = 0$.*

(iii) *$F(z)$ is not an algebraic function of z.*

(*iv*) *$F(z)$ satisfies a functional equation of the form*

$$F(z^q) = \frac{a(z)F(z)+b(z)}{c(z)F(z)+d(z)},$$

where $a(z)$, $b(z)$, $c(z)$, $d(z)$ are polynomials with rational coefficients such that $\triangle(z) = a(z)\,d(z)-b(z)\,c(z)$ does not vanish identically in z. Then if z is an algebraic number satisfying

$$0 < |z| < 1, \quad \triangle(z^{q^\nu}) \neq 0 \quad (\nu = 0, 1, 2, \ldots),$$

$F(z)$ is a transcendental number, but not a Liouville number.

If we apply this theorem to $F(z) = f_k(z)$, then

$$a(z) = 1,\ b(z) = -\frac{z-z^q}{1-z},\ c(z) = 0,\ d(z) = \frac{z-z^q}{1-z},$$

or

$$a(z) = 1,\ b(z) = c(z) = 0,\ d(z) = \frac{1-z^q}{1-z} - z^k,$$

according as to whether $k = 0$ or $1 \leqslant k \leqslant q-1$. We therefore obtain the following result.

THEOREM 3. *Let the case $q = 2$, $k = 1$ be excluded. If z is an algebraic number which satisfies the inequality*

$$0 < |z| < 1 \qquad \textit{for } k = 0,$$

and the inequalities

$$0 < |z| < 1, \ \frac{1-z^{q^\nu}}{1-z^{q^{\nu-1}}} - z^{kq^{\nu-1}} \neq 0 \, (\nu = 1, 2, \ldots) \textit{ for } 1 \leqslant k \leqslant q-1,$$

then $f_k(z)$ is a transcendental number, but not a Liouville number. Furthermore

$$f_k(0) = 1 - \varepsilon_k \qquad (k = 0, 1, \ldots, q-1),$$

and if $k = 1, 2, \ldots, q-1$, $0 < |z| < 1$ and there is a $\nu = 1, 2, \ldots$, such that

$$\frac{1-z^{q^\nu}}{1-z^{q^{\nu-1}}}-z^{kq^{\nu-1}}=0,$$

then $f_k(z)=0$.

5. The zeros of $f_k(z)$. The polynomials

$$\phi_k(z)=\frac{1-z^q}{1-z}-z^k \qquad (k=1,\ 2,\ \ldots,\ q-1)$$

satisfy the functional equations

$$\phi_k(1/z)=z^{-(q-1)}\phi_{q-k-1}(z). \tag{6}$$

Let us assume that $\phi_k(z)$ has $\mu(k)$ zeros of absolute value less than 1, and $\nu(k)$ zeros of absolute value equal to 1. From

$$\phi_{q-1}(z)=1+z+z^2+\ldots+z^{q-2} \qquad (q \text{ arbitrary}),$$
$$\phi_{(q-1)/2}(z)=(1+z+\ldots+z^{(q-3)/2})(1+z^{(q+1)/2}) \qquad (q \text{ odd}),$$

it is clear that

$$\mu(k)=0 \text{ if } k=q-1, \text{ or if } k=(q-1)/2.$$

Further from (6),

$$\nu(k)=\nu(q-k-1). \tag{7}$$

THEOREM 4. *Let* $1\leqslant k\leqslant q-2$ *and* $k\neq(q-1)/2$. *Then* $\mu(k)>0$.

PROOF. The polynomial $\phi_k(z)$ is of exact degree $q-1$; it suffices therefore to prove that $\nu(k)<q-1$. For the product of the zeros of $\phi_k(z)$ is evidently equal to ∓ 1: hence if at least one zero is of absolute value different from 1, then there is also at least one zero of absolute value less than 1.

Since $k\neq(q-1)/2$, it suffices to prove this inequality for $\nu(k)$ if

$$k=1,\ 2,\ \ldots,\ [(q-2)/2].$$

We first note that $\phi_k(z)$ has no multiple zeros on the unit circle. For at such zeros,

$$1-z^q-z^k+z^{k+1}=0,\ qz^{q-1}+kz^{k-1}-(k+1)z^k=0,$$

therefore

$$(q-k)z^q=z^{k+1}-k,$$

whence, by $|z|=1$,

$$q-k\leqslant k+1,\quad k\geqslant(q-1)/2,$$

contrary to hypothesis.

Denote by

$$\zeta=e^{\alpha i},\text{ where } 0<\alpha<2\pi,$$

a zero, hence a simple zero, of

$$\phi_k(z)=1+z+\ldots+z^{q-1}-z^k$$

on the unit circle. Since

$$z^{-\frac{q-1}{2}}\phi_k(z)=\frac{z^{\frac{q}{2}}-z^{-\frac{q}{2}}}{z^{\frac{1}{2}}-z^{-\frac{1}{2}}}-z^{-\frac{q-2k-1}{2}},$$

necessarily

$$\frac{\sin q\alpha/2}{\sin\alpha/2}=\cos\frac{q-2k-1}{2}\alpha-i\sin\frac{q-2k-1}{2}\alpha,$$

and so

$$\sin\frac{q-2k-1}{2}\alpha=0.$$

Hence

$$\alpha=\frac{2n\pi}{q-2k-1},$$

where n is one of the integers $1, 2, \ldots, q-2k-1<q-1$. From this the assertion $\nu(k)<q-1$ follows at once.

Let us combine the last results. We have found:

THEOREM 5. *If* $k = q-1$, *or* $k = (q-1)/2$, *then* $f_k(z)$ *has no zeros inside the unit circle. If* $k = 0$, *then* $f_0(z)$ *has the algebraic zero* $z = 0$, *and all its possible other zeros are transcendental. In all other cases, the zeros of* $f_k(z)$ *are algebraic numbers, and there are an infinity of them inside the unit circle.*

In a similar way, the generating function of integers with more than one missing digit, or with a missing sequence of digits can be investigated.

Manchester University, England.