ON COMPOUND CONVEX BODIES (I)

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THE compounds of a matrix (see e.g. (**1**), chapter 5) play an important role in several branches of mathematics, e.g. in algebraic geometry. The present paper discusses applications of such matrices to the theory of convex bodies and to the geometry of numbers.

For a given dimension n and order p of the compound, it is shown how to associate with every symmetric convex body K in R_n a second symmetric convex body K in R_N where $N = \binom{n}{p}$ is in general greater than n. The bodies K and K are connected by many interesting properties. Thus their volumes satisfy the inequality

$$0 < c_1 \leqslant V(\mathsf{K})V(K)^{-P} \leqslant c_2,$$

where $P = \binom{n-1}{p-1}$, and where c_1 and c_2 depend only on n and p. From this it is deduced that the successive minima $m_1, m_2, \ldots, m_n$ of K, and the successive minima $\mu_1, \mu_2, \ldots, \mu_N$ of K, both for the lattices of all points with integral coordinates, have the property that

$$0 < c_7 M_{\mathrm{K}} \leqslant \mu_{\mathrm{K}} \leqslant M_{\mathrm{K}} \qquad (\mathrm{K} = 1, 2, \ldots, N).$$

Here c_7 depends likewise only on n and p, and $M_1, M_2, \ldots, M_N$ are all the products of p distinct factors m_k arranged according to increasing size. This second result is used to show a general transfer principle connecting systems of linear inequalities with their compound systems.

1. Let $1 \leqslant p \leqslant n-1$, and let

$$X^{(\pi)} = (x_{\pi 1}, x_{\pi 2}, \ldots, x_{\pi n}) \qquad (\pi = 1, 2, \ldots, p)$$

be p points in n-dimensional Euclidean space R_n. There are

$$N = \binom{n}{p}$$

distinct sets of p integers $\nu_1, \nu_2, \ldots, \nu_p$ satisfying

$$1 \leqslant \nu_1 < \nu_2 < \ldots < \nu_p \leqslant n;$$

associate with each such set the determinant

$$x_{\nu_1 \nu_2 \ldots \nu_p} = \begin{vmatrix} x_{1\nu_1} & x_{1\nu_2} & \cdot & \cdot & \cdot & x_{1\nu_p} \\ x_{2\nu_1} & x_{2\nu_2} & \cdot & \cdot & \cdot & x_{2\nu_p} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{p\nu_1} & x_{p\nu_2} & \cdot & \cdot & \cdot & x_{p\nu_p} \end{vmatrix}.$$

Finally arrange these determinants in an arbitrary order (e.g. lexicographically) and denote them in this order by $\xi_1, \xi_2,..., \xi_N$. There corresponds then to the set of points $X^{(1)}, X^{(2)},..., X^{(p)}$ in R_n the point

$$\Xi = (\xi_1, \xi_2,..., \xi_N), \quad = [X^{(1)}, X^{(2)},..., X^{(p)}] \text{ say,}$$

in N-dimensional Euclidean space R_N.

If, in particular, $X^{(1)}, X^{(2)},..., X^{(p)}$ are linearly independent, the $p \times n$ matrix

$$\begin{bmatrix} x_{11} & x_{12} & . & . & . & x_{1n} \\ x_{21} & x_{22} & . & . & . & x_{2n} \\ . & . & . & . & . & . \\ x_{p1} & x_{p2} & . & . & . & x_{pn} \end{bmatrix}$$

is of exact rank p, and so at least one of the minors $x_{\nu_1 \nu_2 ... \nu_p}$ is not zero. Hence Ξ is in this case different from the origin O of R_N. On the other hand $\Xi = O$ if the given points in R_n are linearly dependent, e.g. if two of them coincide.

It is well known (see e.g. (**3**), chapter 5) that if $2 \leqslant p \leqslant n-2$, the determinants $x_{\nu_1 \nu_2 ... \nu_p}$ cannot assume values independent of one another, but satisfy a certain set of homogeneous quadratic equations; e.g. in the lowest non-trivial case when $n = 4$, $p = 2$ there is just one such condition, and, on changing the sign of one of the determinants, it can be written as

$$\xi_1 \xi_4 + \xi_2 \xi_5 + \xi_3 \xi_6 = 0.$$

In other words, for all choices of $X^{(1)}, X^{(2)},..., X^{(p)}$ in R_n the derived point $\Xi = [X^{(1)}, X^{(2)},..., X^{(p)}]$ is restricted to a certain algebraic manifold $\mathbf{\Omega}(n,p)$ in R_N in the form of a cone of centre O, the *Grassmann manifold*, and this manifold coincides with the whole space only when either $p = 1$ or $p = n-1$.

2. Let now $K^{(1)}, K^{(2)},..., K^{(p)}$ be any p *bounded closed convex bodies in* R_n. To simplify the discussion, and because this suffices for the later application, we shall impose the further condition that *each body $K^{(\pi)}$ contains the origin O of the coordinate system as an inner point and is, moreover, symmetric in this point*. It is *not* demanded that the p bodies $K^{(1)}, K^{(2)},..., K^{(p)}$ are all distinct, and in fact these bodies will later on be made to coincide.

Denote now by

$$\Sigma = \langle K^{(1)}, K^{(2)},..., K^{(p)} \rangle$$

the set of all points $\Xi = [X^{(1)}, X^{(2)},..., X^{(p)}]$ where, for $\pi = 1, 2,..., p$, $X^{(\pi)}$ runs independently over all points of $K^{(\pi)}$. From this definition it is at once obvious that Σ is a bounded closed point set which lies entirely on the manifold $\mathbf{\Omega}(n,p)$. In general, Σ naturally need not be a convex set.

Denote then by

$$\mathsf{K} = [K^{(1)}, K^{(2)},..., K^{(p)}]$$

the *convex hull of* Σ, i.e. the smallest closed convex set that contains Σ. We call K the *compound of* $K^{(1)}, K^{(2)},..., K^{(p)}$.

Since the origin O of R_n belongs to all sets $K^{(\pi)}$, the compound K similarly contains the origin O of R_N. Moreover, K is symmetrical in O since $K^{(1)}$ is symmetrical in O and

$$[-X^{(1)}, X^{(2)},..., X^{(p)}] = -[X^{(1)}, X^{(2)},..., X^{(p)}].$$

We can further show that O is an inner point of K, and hence that the compound is a convex *body*. For, by the hypothesis, O is an inner point of each body $K^{(\pi)}$. Hence a positive number δ can be chosen such that the closed sphere $|X| \leqslant \delta$ is a subset of each of $K^{(1)}, K^{(2)},..., K^{(p)}$. Then the points

$$P_1 = (\delta, 0, 0,..., 0),\ P_2 = (0, \delta, 0,..., 0),\quad ...,\quad P_n = (0, 0, 0,..., \delta)$$

on the coordinate axes and their images in O are elements of all bodies $K^{(\pi)}$, and therefore the derived points

$$\pm[P_{\nu_1}, P_{\nu_2},..., P_{\nu_p}] \quad \text{where } 1 \leqslant \nu_1 < \nu_2 < ... < \nu_p \leqslant n$$

belong to Σ. But these derived points are exactly all the points on the coordinate axes in R_N of distance δ^p from the origin, and their convex hull is the generalized octahedron T consisting of all points Ξ for which

$$|\xi_1|+|\xi_2|+...+|\xi_N| \leqslant \delta^p.$$

Evidently T contains O as an inner point and is itself contained in K, whence the assertion.

We note that the compound $\mathsf{K} = [K^{(1)}, K^{(2)},..., K^{(p)}]$ obviously does not depend on the order of $K^{(1)}, K^{(2)},..., K^{(p)}$, and that, in fact, this is the case even when only a single one of these bodies is symmetrical in the origin.

3. Let $X \to X' = \Omega X$, or in explicit form

$$x_h \to x'_h = \sum_{k=1}^{n} \omega_{hk} x_k \qquad (h = 1, 2,..., n),$$

be a non-singular affine transformation of R_n into itself. Thus the determinant, ω say, of the transformation matrix $\Omega = (\omega_{hk})$ does not vanish. Such a transformation Ω changes every bounded, closed, symmetric, convex body K in R_n into a body $K' = \Omega K$ of the same kind. If the letter V is used to denote the volume of a body, clearly

$$V(K') = V(\Omega K) = |\omega| V(K).$$

The transformation Ω of R_n generates in R_N a likewise affine transformation, the pth compound $\Omega^{(p)}$ of Ω. This compound is defined as follows. Let $X^{(1)}, X^{(2)},..., X^{(p)}$ be any p points in R_n, and let

$$\Xi = (\xi_1, \xi_2,..., \xi_N) = [X^{(1)}, X^{(2)},..., X^{(p)}]$$

be the corresponding point in R_N. On applying Ω simultaneously to all $X^{(\pi)}$, a second point

$$\Xi' = (\xi'_1, \xi'_2,..., \xi'_N) = [\Omega X^{(1)}, \Omega X^{(2)},..., \Omega X^{(p)}]$$

in R_N is obtained which may be denoted symbolically by

$$\Xi' = \Omega^{(p)}\,\Xi.$$

Here $\Omega^{(p)}$ again represents an affine transformation of the space R_N into itself. The matrix

$$\Omega^{(p)} = (\omega^{(p)}_{\mathrm{HK}}) \qquad (\mathrm{H, K} = 1, 2, \ldots, N)$$

of this transformation has as its elements the N^2 minors of order p of the original matrix $\Omega = (\omega_{hk})$, both indices H and K being arranged in the same order as in § 1 when defining the order of the coordinates of Ξ. It is shown in determinant theory that the determinant of $\Omega^{(p)}$, $\omega^{(p)}$ say, is given by

$$\omega^{(p)} = \omega^P \quad \text{where } P = \binom{n-1}{p-1}.$$

Hence the compound transformation $\Omega^{(p)}$ is likewise non-singular.

The transformation $\Xi \rightarrow \Xi' = \Omega^{(p)}\,\Xi$ changes Σ and K into new sets $\Sigma' = \Omega^{(p)}\,\Sigma$ and $\mathsf{K}' = \Omega^{(p)}\,\mathsf{K}$ which may be expressed explicitly in the form

$$\Sigma' = \langle \Omega K^{(1)}, \Omega K^{(2)}, \ldots, \Omega K^{(p)} \rangle \quad \text{and} \quad \mathsf{K}' = [\Omega K^{(1)}, \Omega K^{(2)}, \ldots, \Omega K^{(p)}].$$

This is obvious in the case of Σ', and is for K' due to the fact that every affine transformation changes the convex hull of a set into the convex hull of the transformed set.

By the value of the determinant of $\Omega^{(p)}$, it is again clear that the volumes of the compound bodies K and $\mathsf{K}' = \Omega^{(p)}\,\mathsf{K}$ are connected by the formula

$$V(\mathsf{K}') = V(\Omega^{(p)}\,\mathsf{K}) = |\omega|^P\, V(\mathsf{K}).$$

4. In this and the next sections we shall only be concerned with the special case when the convex bodies $K^{(1)}, K^{(2)}, \ldots, K^{(p)}$ defining

$$\mathsf{K} = [K^{(1)}, K^{(2)}, \ldots, K^{(p)}]$$

are identical: $K^{(1)} = K^{(2)} = \ldots = K^{(p)} = K$, say. We then write $\mathsf{K} = [K]^{(p)}$, and similarly $\mathsf{K}' = \Omega^{(p)}\mathsf{K} = [\Omega K]^{(p)}$. The correspondence $K \rightarrow \mathsf{K} = [K]^{(p)}$ gives now a mapping of the set of all closed, bounded, symmetric, convex bodies in R_n into the set of all analogous bodies in R_N.

We begin with some remarks on spheres and ellipsoids. Let

$$G_n\colon\ |X| \leqslant 1$$

be the unit sphere in R_n, and let

$$\Gamma^{(p)}_n = [G_n]^{(p)}$$

be its compound in R_N. In general, $\Gamma^{(p)}_n$ is not a sphere; it has, however, interesting symmetry properties and may deserve a detailed study on its own account.

Next let E be any bounded closed ellipsoid in R_n with centre at O, and let $\mathsf{E} = [E]^{(p)}$ be its pth compound. By the theory of such ellipsoids there exists an affine transformation $X \to \Omega X$ of R_n into itself, of determinant $\omega \neq 0$, such that $E = \Omega G_n$; hence

$$V(E) = |\omega| V(G_n).$$

Further also $\mathsf{E} = \Omega^{(p)}\Gamma_n^{(p)}$ and therefore

$$V(\mathsf{E}) = |\omega|^P\, V(\Gamma_n^{(p)}),$$

where again $P = \binom{n-1}{p-1}$. The product

$$V(\mathsf{E})V(E)^{-P} = V(\Gamma_n^{(p)})V(G_n)^{-P} > 0$$

is therefore independent of the special ellipsoid E and a function only of n and p.

5. We can now prove the first main result.

THEOREM 1. *There exist two positive constants c_1 and c_2 with $c_1 < c_2$ and depending only on n and p, with the following property.*

If K is any closed bounded symmetric convex body in R_n, and if $\mathsf{K} = [K]^{(p)}$ is its p-th compound in R_N, then

$$c_1 \leqslant V(\mathsf{K})V(K)^{-P} \leqslant c_2, \quad \textit{where } P = \binom{n-1}{p-1}.$$

Proof. Let E be the ellipsoid with centre at O which is circumscribed to K and of smallest volume. A theorem due to John (**4**) states that there exists a second ellipsoid $n^{-\frac{1}{2}}E$ obtained from E by the similarity transformation $X \to n^{-\frac{1}{2}}X$ which is inscribed in K. Thus

$$n^{-\frac{1}{2}}E \subseteq K \subseteq E;$$

hence
$$V(n^{-\frac{1}{2}}E) = n^{-\frac{1}{2}n}V(E) \leqslant V(K) \leqslant V(E). \tag{1}$$

Let now $\mathsf{K} = [K]^{(p)}$ and $\mathsf{E} = [E]^{(p)}$ be the compounds of K and E. Then also
$$[n^{-\frac{1}{2}}E]^{(p)} = n^{-\frac{1}{2}p}\,\mathsf{E},$$

because the pth compound of the affine transformation $X \to n^{-\frac{1}{2}}X$ is given by $\Xi \to (n^{-\frac{1}{2}})^p\, \Xi$, as follows at once from the definition of $\Omega^{(p)}$. Further

$$V(n^{-\frac{1}{2}p}\,\mathsf{E}) = (n^{-\frac{1}{2}p})^N\, V(\mathsf{E}) = (n^{-\frac{1}{2}n})^P\, V(\mathsf{E}).$$

Next, it is evident from the definition that $K_1 \subseteq K_2$ implies that also $[K_1]^{(p)} \subseteq [K_2]^{(p)}$. Therefore

$$n^{-\frac{1}{2}p}\,\mathsf{E} \subseteq \mathsf{K} \subseteq \mathsf{E},$$

whence
$$V(n^{-\frac{1}{2}p}\,\mathsf{E}) = (n^{-\frac{1}{2}n})^P\, V(\mathsf{E}) \leqslant V(\mathsf{K}) \leqslant V(\mathsf{E}). \tag{2}$$

On combining now (1) and (2), it follows that

$$(n^{-\frac{1}{2}n})^P\, V(\mathsf{E})V(E)^{-P} \leqslant V(\mathsf{K})V(K)^{-P} \leqslant (n^{+\frac{1}{2}n})^P\, V(\mathsf{E})V(E)^{-P}.$$

Here, by the last section,

$$V(\mathsf{E})V(E)^{-P} = V(\Gamma_n^{(p)})V(G_n)^{-P}$$

is a number depending only on n and p, and so the assertion holds with the constants

$$c_1 = n^{-\frac{1}{2}nP}\, V(\Gamma_n^{(p)})V(G_n)^{-P} \quad \text{and} \quad c_2 = n^{+\frac{1}{2}nP}\, V(\Gamma_n^{(p)})V(G_n)^{-P}.$$

It would be of interest to find the best possible values for c_1 and c_2, and to decide for which bodies these bounds are attained.

In the second part of this paper, I deal with the question of how far Theorem 1 can be extended to general compounds $[K^{(1)}, K^{(2)},..., K^{(p)}]$.

6. For the applications to the geometry of numbers it is useful to determine the distance function of a compound body.

It is well known that every closed, bounded, symmetric, convex body K in R_n has a distance function $F(X)$ such that K consists exactly of all points X for which $F(X) \leqslant 1$. Here a distance function is a real-valued function $F(X) = F(x_1, x_2,..., x_n)$ of X in R_n with the following properties.

(a) $$F(X) > 0 \quad \text{if } X \neq O; \quad F(O) = 0,$$

(b) $$F(tX) = |t|F(X) \quad \text{for real } t,$$

(c) $$F(X+Y) \leqslant F(X)+F(Y).$$

Similar distance functions, but with Ξ as the variable, naturally exist for the convex bodies in R_N.

Let now again $K^{(1)}, K^{(2)},..., K^{(p)}$ be p bounded, closed, symmetric, convex bodies in R_n, and let $\mathsf{K} = [K^{(1)}, K^{(2)},..., K^{(p)}]$ be their compound in R_N. Further denote by $F^{(\pi)}(X)$, for $\pi = 1, 2,..., p$, the distance function of $K^{(\pi)}$, and by $\Phi(\Xi)$ the distance function of K. Our problem is to express $\Phi(\Xi)$ in terms of $F^{(1)}(X), F^{(2)}(X),..., F^{(p)}(X)$. We shall solve this problem in the next sections.

7. Every point Ξ in R_N can be written in many ways as a finite sum

$$\Xi = \sum_{\rho=1}^{r} [X_\rho^{(1)}, X_\rho^{(2)},..., X_\rho^{(p)}], \tag{1}$$

where the $X_\rho^{(\pi)}$ are suitable points in R_n, and r can be arbitrary. For the unit points on the coordinate axes in R_N certainly admit such a representation, even as a sum of one single term. The same is therefore true for all points on these axes and so, by vector addition, for all points Ξ in R_N.

Denote, as usual, by $|X|$ the length of $X = (x_1, x_2,..., x_p)$,

$$|X| = +(x_1^2+x_2^2+...+x_n^2)^{\frac{1}{2}},$$

and similarly by $|\Xi|$ the length of $\Xi = (\xi_1, \xi_2,..., \xi_N)$,

$$|\Xi| = +(\xi_1^2+\xi_2^2+...+\xi_N^2)^{\frac{1}{2}}.$$

Every coordinate of the point

$$[X_\rho^{(1)}, X_\rho^{(2)},..., X_\rho^{(p)}]$$

is a minor of the corresponding $p \times n$ matrix. There exists then a positive constant c_3 depending only on n and p such that

$$|[X_\rho^{(1)}, X_\rho^{(2)},..., X_\rho^{(p)}]| \leqslant c_3 |X_\rho^{(1)}| |X_\rho^{(2)}| ... |X_\rho^{(p)}|.$$

Hence the representation (1) of Ξ implies that

$$|\Xi| \leqslant c_3 \sum_{\rho=1}^{r} |X_\rho^{(1)}||X_\rho^{(2)}|\ldots|X_\rho^{(p)}|. \tag{2}$$

Thus, if $\Xi \neq O$, then not all points $X_\rho^{(\pi)}$ can be too near to the origin.

8. We define now a function $\Psi(\Xi)$ as the lower bound

$$\Psi(\Xi) = \inf \sum_{\rho=1} F^{(1)}(X_\rho^{(1)})F^{(2)}(X_\rho^{(2)})\ldots F^{(p)}(X_\rho^{(p)}) \tag{1}$$

extended over all finite decompositions

$$\Xi = \sum_{\rho=1}^{r} [X_\rho^{(1)}, X_\rho^{(2)},\ldots, X_\rho^{(p)}] \tag{2}$$

of Ξ, $F^{(\pi)}(X)$ having the same meaning as in § 6.

The function $\Psi(\Xi)$ is properly defined in this way because Ξ always admits at least one decomposition. It is obvious that $\Psi(\Xi)$ is always non-negative, and that $\Psi(O) = 0$ since

$$O = [O, O, \ldots, O].$$

We next show that $\Psi(\Xi) > 0$ if $\Xi \neq O$. By a classical property of convex distance functions, a positive constant γ_1 can be chosen such that

$$F^{(\pi)}(X) \geqslant \gamma_1 |X| \text{ for all } X \qquad (\pi = 1, 2,\ldots, p). \tag{3}$$

By the last section, the decomposition (2) of Ξ implies that

$$|\Xi| \leqslant c_3 \sum_{\rho=1}^{r} |X_\rho^{(1)}||X_\rho^{(2)}|\ldots|X_\rho^{(p)}|,$$

while, by (3),

$$\sum_{\rho=1}^{r} F^{(1)}(X_\rho^{(1)})F^{(2)}(X_\rho^{(2)})\ldots F^{(p)}(X_\rho^{(p)}) \geqslant \gamma_1^p \sum_{\rho=1}^{p} |X_\rho^{(1)}||X_\rho^{(2)}|\ldots|X_\rho^{(p)}|.$$

It follows therefore that always

$$\Psi(\Xi) \geqslant \gamma_2 |\Xi|, \quad \text{where } \gamma_2 = \gamma_1^p / c_3, \tag{4}$$

whence the assertion.

Furthermore, if Ξ admits any decomposition (2), then $t\,\Xi$ has the derived decomposition

$$t\,\Xi = \sum_{\rho=1}^{r} [tX_\rho^{(1)}, X_\rho^{(2)},\ldots, X_\rho^{(p)}],$$

and vice versa; hence

$$\Psi(t\,\Xi) = |t|\Psi(\Xi) \tag{5}$$

since $F^{(1)}(tX_\rho^{(1)}) = |t|F^{(1)}(X_\rho^{(1)})$.

Finally, $\Psi(\Xi)$ satisfies the triangle inequality

$$\Psi(\Xi + \mathrm{H}) \leqslant \Psi(\Xi) + \Psi(\mathrm{H}). \tag{6}$$

For let $\epsilon > 0$ be arbitrarily small; then two decompositions

$$\Xi = \sum_{\rho=1}^{r} [X_\rho^{(1)}, X_\rho^{(2)},..., X_\rho^{(p)}] \quad \text{and} \quad \mathsf{H} = \sum_{\sigma=1}^{s} [Y_\sigma^{(1)}, Y_\sigma^{(2)},..., Y_\sigma^{(p)}]$$

of Ξ and H can be chosen such that

$$\Psi(\Xi) > \sum_{\rho=1}^{r} F^{(1)}(X_\rho^{(1)})...F^{(p)}(X_\rho^{(p)}) - \tfrac{1}{2}\epsilon$$

and
$$\Psi(\mathsf{H}) > \sum_{\sigma=1}^{s} F^{(1)}(Y_\sigma^{(1)})...F^{(p)}(Y_\sigma^{(p)}) - \tfrac{1}{2}\epsilon.$$

Since now

$$\Xi + \mathsf{H} = \sum_{\rho=1}^{r} [X_\rho^{(1)}, X_\rho^{(2)},..., X_\rho^{(p)}] + \sum_{\sigma=1}^{s} [Y_\sigma^{(1)}, Y_\sigma^{(2)},..., Y_\sigma^{(p)}],$$

we find that
$$\Psi(\Xi+\mathsf{H}) < \{\Psi(\Xi) + \tfrac{1}{2}\epsilon\} + \{\Psi(\mathsf{H}) + \tfrac{1}{2}\epsilon\},$$

whence the assertion when ϵ tends to zero.

The formulae (4), (5), and (6), together with $\Psi(O) = 0$, mean that $\Psi(\Xi)$ is a convex distance function.

9. It will now be proved that $\Psi(\Xi)$ is in fact the distance function of $\mathsf{K} = [K^{(1)}, K^{(2)},..., K^{(p)}]$, i.e. that $\Psi(\Xi) \equiv \Phi(\Xi)$. This proof consists of two parts; for it has to be shown that, if Ξ is any point of K, then $\Psi(\Xi) \leqslant 1$, and that the converse of this statement is also true.

(i) Let Ξ be an arbitrary point of K. Since K is the convex hull of the set Σ, there exist (see (**2**), p. 9) $r = N+1$ points of Σ, the points $\Xi_1, \Xi_2,..., \Xi_r$ say, such that Ξ is an inner or boundary point of the simplex with vertices at the points Ξ_ρ. Thus Ξ can be written as

$$\Xi = \sum_{\rho=1}^{r} t_\rho \Xi_\rho,$$

where $t_1, t_2,..., t_r$ are real numbers such that

$$t_1 \geqslant 0, \quad t_2 \geqslant 0, \quad ..., \quad t_r \geqslant 0, \qquad \sum_{\rho=1}^{r} t_\rho = 1.$$

By the definition of Ξ, each point Ξ_ρ can be expressed in the form

$$\Xi_\rho = [\bar{X}_\rho^{(1)}, X_\rho^{(2)},..., X_\rho^{(p)}] \qquad (\rho = 1, 2,..., r),$$

where
$$\bar{X}_\rho^{(1)} \in K^{(1)}, \quad X_\rho^{(2)} \in K^{(2)}, \quad ..., \quad X_\rho^{(p)} \in K^{(p)}$$

and therefore

$$F^{(1)}(\bar{X}_\rho^{(1)}) \leqslant 1, \qquad F^{(2)}(X_\rho^{(2)}) \leqslant 1, \quad ..., \quad F^{(p)}(X_\rho^{(p)}) \leqslant 1.$$

Put now
$$X_\rho^{(1)} = t_\rho \bar{X}_\rho^{(1)}, \quad \text{so that } F^{(1)}(X_\rho^{(1)}) \leqslant t_\rho.$$

Then
$$\Xi = \sum_{\rho=1}^{r} [X_\rho^{(1)}, X_\rho^{(2)},..., X_\rho^{(p)}]$$

and
$$\sum_{\rho=1}^{r} F^{(1)}(X_\rho^{(1)}) F^{(2)}(X_\rho^{(2)})...F^{(p)}(X_\rho^{(p)}) \leqslant \sum_{\rho=1}^{r} t_\rho = 1,$$

whence $\Psi(\Xi) \leqslant 1$ by the definition of this function.

(ii) To prove also the converse, assume first that the point Ξ satisfies the stronger inequality $\Psi(\Xi) < 1$, and choose a positive number ϵ such that also $\Psi(\Xi)+\epsilon \leqslant 1$. Further select a decomposition

$$\Xi = \sum_{\rho=1}^{r} [\bar{X}_\rho^{(1)}, \bar{X}_\rho^{(2)},..., \bar{X}_\rho^{(p)}]$$

of Ξ for which

$$\sum_{\rho=1}^{r} F^{(1)}(\bar{X}_\rho^{(1)})F^{(2)}(\bar{X}_\rho^{(2)})...F^{(p)}(\bar{X}_\rho^{(p)}) \leqslant \Psi(\Xi)+\epsilon \leqslant 1.$$

There is no loss of generality in assuming that none of the points $\bar{X}_\rho^{(\pi)}$ lies at the origin. The numbers

$$\tau_\rho^{(\pi)} = F^{(\pi)}(\bar{X}_\rho^{(\pi)}) \quad (\pi = 1, 2,..., p, \quad \rho = 1, 2,..., r)$$

are thus all positive, and each point $\bar{X}_\rho^{(\pi)}$ is of the form

$$\bar{X}_\rho^{(\pi)} = \tau_\rho^{(\pi)} X_\rho^{(\pi)} \quad \text{where } F^{(\pi)}(X_\rho^{(\pi)}) = 1.$$

Put now
$$t_\rho = \tau_\rho^{(1)} \tau_\rho^{(2)}...\tau_\rho^{(p)} \qquad (\rho = 1, 2,..., r),$$

so that t_ρ is likewise positive. Then

$$\Xi = \sum_{\rho=1}^{r} t_\rho [X_\rho^{(1)}, X_\rho^{(2)},..., X_\rho^{(p)}], \tag{1}$$

and here
$$\sum_{\rho=1}^{r} t_\rho = \sum_{\rho=1}^{r} F^{(1)}(\bar{X}_\rho^{(1)})F^{(2)}(\bar{X}_\rho^{(2)})...F^{(p)}(\bar{X}_\rho^{(p)}) \leqslant 1.$$

Further
$$X_\rho^{(1)} \in K^{(1)}, \quad X_\rho^{(2)} \in K^{(2)}, \quad ..., \quad X_\rho^{(p)} \in K^{(p)},$$

and therefore $[X_\rho^{(1)}, X_\rho^{(2)},..., X_\rho^{(p)}] \in \Sigma$. Since also $O \in \Sigma$, it follows then from (1) that Ξ belongs to the convex hull of Σ, i.e. to K.

This proof assumed that $\Psi(\Xi) < 1$. But K is a closed set, and $\Psi(\Xi)$ is a distance function, hence is continuous. Therefore the less strong assumption that $\Psi(\Xi) \leqslant 1$ still implies that Ξ belongs to K. This concludes the proof.

From now on we use the notation $\Phi(\Xi)$ for the distance function of K. It is implicit in the last proof that $\Phi(\Xi)$ may also be defined by

$$\Phi(\Xi) = \min \sum_{\rho=1}^{N+1} F^{(1)}(X_\rho^{(1)})F^{(2)}(X_\rho^{(2)})...F^{(p)}(X_\rho^{(p)}),$$

where the minimum is now extended only over decompositions

$$\Xi = \sum_{\rho=1}^{N+1} [X_\rho^{(1)}, X_\rho^{(2)},..., X_\rho^{(p)}]$$

of Ξ into $r = N+1$ terms. By means of Weierstrass's theorem one shows easily that the minimum is attained. But as we make no use of this result, the proof may be omitted.

10. The results so far obtained will be applied to the geometry of numbers. We begin by defining *compound lattices*.

Let L be any n-dimensional lattice in R_n, say of basis $Z_1, Z_2,..., Z_n$ and of determinant $d(L) = |\{Z_1, Z_2,..., Z_n\}|$. Here the symbol $\{Z_1, Z_2,..., Z_n\}$ denotes the determinant of the n base points. The general point of L is then of the form $X = u_1 Z_1 + u_2 Z_2 + ... + u_n Z_n$ where $u_1, u_2,..., u_n$ run over all integers.

Assume $X^{(1)}, X^{(2)},..., X^{(p)}$ describe separately all the points of L. The compound points $\Xi = [X^{(1)}, X^{(2)},..., X^{(p)}]$ form a certain point set Π situated on the Grassmann manifold $\mathbf{\Omega}(n,p)$ in R_N which, in general, is not itself a lattice. However, a unique N-dimensional lattice Λ in R_N may be derived from Π as the set consisting of all finite sums

$$\Xi = \sum_{\rho=1}^{r} \Xi_\rho,$$

where the points Ξ_ρ run separately over the elements of Π. We call Λ the pth compound of L.

We must show that the compound so defined is in fact a lattice, and begin with a special case. Let L_0 be the lattice of all points in R_n with integral coordinates; this lattice has the basis

$$Z_1 = (1,0,...,0), \quad Z_2 = (0,1,...,0), \quad ..., \quad Z_n = (0,0,...,1)$$

and the determinant $d(L_0) = 1$. It is obvious that its compound lattice, Λ_0 say, contains only points with integral coordinates. In fact, Λ_0 is identical with the lattice of all points in R_N with integral coordinates. For the N compound points $[Z_{\nu_1}, Z_{\nu_2},..., Z_{\nu_p}]$, where

$$1 \leqslant \nu_1 < \nu_2 < ... < \nu_p \leqslant n,$$

form exactly all the N distinct unit points on the coordinate axes in R_N, i.e. the points with one coordinate equal to 1 and the others equal to 0. Also the negative unit points can be written in a similar form as compounds of the Z's. The assertion is thus a consequence of the obvious fact that every point with integral coordinates may be expressed as a sum of finitely many positive and negative unit points.

It is now easy to show that also in the general case the compound set Λ is a lattice. There exists to the given lattice L in R_n an affine transformation $X \to X' = \Omega X$ such that $L = \Omega L_0$; let ω be its determinant. Then $d(L) = |\omega| d(L_0)$ and therefore

$$\omega = \pm d(L).$$

Now Ω generates in R_N the compound affine transformation

$$\Xi \to \Xi' = \Omega^{(p)}\Xi$$

of determinant $\omega^{(p)} = \omega^P = \pm d(L)^P$. This transformation $\Omega^{(p)}$ evidently changes the special compound lattice Λ_0 corresponding to L_0 into the compound Λ corresponding to L. But then Λ is likewise a lattice because the image of every lattice under any non-singular affine transformation is again a lattice. Further $d(\Lambda) = |\omega^{(p)}| d(\Lambda_0)$, and so Λ has the determinant

$$d(\Lambda) = d(L)^P.$$

The construction of the pth compound just given associates with every lattice L in R_n a unique lattice Λ in R_N. We note that, on the other hand, if $2 \leqslant p \leqslant n-2$, not every lattice Λ in R_N can be obtained as the pth compound of some lattice L in R_n. For the lattice Λ may be chosen such that the N^2 coordinates of its N base points are algebraically independent real numbers. Then no point of Λ distinct from O lies on the Grassmann manifold $\mathbf{\Omega}(n, p)$ because this manifold is defined by homogeneous quadratic equations with rational coefficients. But, by the definition, a compound lattice is always generated by its points on $\mathbf{\Omega}(n, p)$.

11. Let K be again a bounded, closed, symmetric, convex body in R_n, and let $\mathsf{K} = [K]^{(p)}$ be its pth compound body in R_N. There is some interest in comparing the number-geometrical properties of K with those of K. A few such properties will now be considered.

One basic functional in the geometry of numbers is the lattice determinant $\Delta(K)$ of a body K; it is defined as the lower bound of the determinants $d(L)$ of all K-admissible lattices L. Here L is said to be K-admissible if none of its points distinct from O is an *inner point* of K. The lattice determinant $\Delta(\mathsf{K})$ is defined in an analogous way; note that in its case the lower bound is extended over *all* K-admissible lattices, not only the compound ones.

Minkowski's classical theorem on convex bodies is equivalent to the inequality

$$2^n \Delta(K) \geqslant V(K).$$

Another well-known theorem of his, which was first proved by E. Hlawka, states that

$$V(K) \geqslant 2\zeta(n)\Delta(K) \qquad \left(\zeta(n) = \sum_{l=1}^{\infty} l^{-n}\right).$$

Similar inequalities

$$2^N \Delta(\mathsf{K}) \geqslant V(\mathsf{K}) \geqslant 2\zeta(N)\Delta(\mathsf{K})$$

hold, of course, for the compound body. Therefore Theorem 1 at once leads to the following result.

THEOREM 2. *There exist two positive constants c_4 and c_5, with $c_4 < c_5$ and depending only on n and p, with the following property.*

If K is any closed, bounded, symmetric, convex body in R_n, and if $\mathsf{K} = [K]^{(p)}$ is its p-th compound in R_N, then

$$c_4 \leqslant \Delta(\mathsf{K})\Delta(K)^{-P} \leqslant c_5, \quad \textit{where } P = \binom{n-1}{p-1}.$$

Remark. One can define a second functional

$$\Delta^*(\mathsf{K}) = \inf d(\Lambda),$$

where now the lower bound extends only over those K-admissible lattices Λ in R_N that are pth compounds of lattices L in R_n. Then it may be proved that $\Delta(\mathsf{K})$ and $\Delta^*(\mathsf{K})$ satisfy the inequality

$$\Delta(\mathsf{K}) \leqslant \Delta^*(\mathsf{K}) \leqslant c_6\,\Delta(\mathsf{K}),$$

where $c_6 > 0$ again depends only on n and p. Hence Theorem 2 remains valid, but with different constants, if in it $\Delta(\mathsf{K})$ is replaced by $\Delta^*(\mathsf{K})$.

12. Let K and $\mathsf{K} = [K]^{(p)}$ have the same meaning as before; let $F(X)$ and $\Phi(\Xi)$ be the distance functions of K and K, respectively, and let L be a lattice in R_n and Λ its pth compound in R_N.

A well-known general theorem of Minkowski deals with the successive minima of K in L. These minima are defined as follows.

There exists a point $X_1 \neq O$ in L such that $F(X_1) = m_1 = m_1(K, L)$ is a minimum; m_1 is called the *first minimum* of K in L. Next let $2 \leqslant k \leqslant n$, and assume that the points $X_1, X_2,..., X_{k-1}$ in L and the corresponding successive minima

$$F(X_h) = m_h = m_h(K, L) \qquad (h = 1, 2,..., k-1)$$

have already been defined. Then there exists a point X_k in L linearly independent of $X_1, X_2,..., X_{k-1}$ for which $F(X_k) = m_k = m_k(K, L)$ is as small as possible; m_k is called the *k-th minimum* of K in L. Thus the n lattice points $X_1, X_2,..., X_n$ are linearly independent, and the successive minima satisfy the inequalities

$$0 < m_1 \leqslant m_2 \leqslant ... \leqslant m_n < \infty.$$

These minima also satisfy the following property. *If $Y_1, Y_2,..., Y_n$ are any n independent points of L ordered such that*

$$F(Y_1) \leqslant F(Y_2) \leqslant ... \leqslant F(Y_n),$$

then $$F(Y_1) \geqslant m_1, \quad F(Y_2) \geqslant m_2, \quad ..., \quad F(Y_n) \geqslant m_n.$$

In the last chapter of his *Geometrie der Zahlen*, Minkowski proved the fundamental inequalities

$$2^n(n!)^{-1}d(L) \leqslant m_1 m_2 ... m_n V(K) \leqslant 2^n d(L) \tag{1}$$

which contain his theorem $V(K) \leqslant 2^n\Delta(K)$ as an obvious consequence.

Naturally these results have their analogues with respect to the compound body K and the compound lattice Λ. There exist N linearly independent points $\Xi_1, \Xi_2,..., \Xi_N$ in Λ generating the successive minima

$$\Phi(\Xi_{\mathrm{K}}) = \mu_{\mathrm{K}} = \mu_{\mathrm{K}}(\mathsf{K}, \Lambda) \qquad (\mathrm{K} = 1, 2,..., N)$$

of K in Λ, and these satisfy the inequalities

$$2^N(N!)^{-1}\,d(\Lambda) \leqslant \mu_1\mu_2\ldots\mu_N\,V(\mathsf{K}) \leqslant 2^N d(\Lambda). \tag{2}$$

Also if $\mathsf{H}_1, \mathsf{H}_2,\ldots, \mathsf{H}_N$ are N linearly independent points of Λ arranged so that

$$\Phi(\mathsf{H}_1) \leqslant \Phi(\mathsf{H}_2) \leqslant \ldots \leqslant \Phi(\mathsf{H}_N),$$

then

$$\Phi(\mathsf{H}_1) \geqslant \mu_1, \quad \Phi(\mathsf{H}_2) \geqslant \mu_2, \quad \ldots, \quad \Phi(\mathsf{H}_N) \geqslant \mu_N. \tag{3}$$

13. Our next aim will be to find relations connecting the two sets of minima $m_k(K, L)$ and $\mu_{\mathrm{K}}(\mathsf{K}, \Lambda)$. This work will be based on the inequality

$$c_1 \leqslant V(\mathsf{K})V(K)^{-P} \leqslant c_2 \tag{1}$$

of Theorem 1 and on the equation

$$d(\Lambda) = d(L)^P \tag{2}$$

which connects the determinants of L and Λ.

From this equation, and from the two formulae (1) and (2) of the last section, it follows immediately that

$$\frac{2^{N-nP}}{N!} \leqslant \frac{\mu_1\mu_2\ldots\mu_N\,V(\mathsf{K})}{\{m_1 m_2\ldots m_n\,V(K)\}^P} \leqslant 2^{N-nP}(n!)^P.$$

Therefore, by (1), there exist two positive constants c_7 and c_8 depending only on n and p and such that $c_7 < c_8$ and

$$c_7(m_1 m_2\ldots m_n)^P \leqslant \mu_1\mu_2\ldots\mu_N \leqslant c_8(m_1 m_2\ldots m_n)^P. \tag{3}$$

We have thus obtained an inequality in which the only variables occurring are the successive minima of the two bodies. As will be proved, this single inequality can be replaced by a set of inequalities, one for each of the μ's.

14. Form the N products

$$M_{\nu_1\nu_2\ldots\nu_p} = m_{\nu_1} m_{\nu_2}\ldots m_{\nu_p},$$

where $\nu_1, \nu_2,\ldots, \nu_p$ run over all sets of p indices such that

$$1 \leqslant \nu_1 < \nu_2 < \ldots < \nu_p \leqslant n.$$

We arrange these products in order of increasing size and rename them then $M_1, M_2,\ldots, M_N$; thus

$$0 < M_1 \leqslant M_2 \leqslant \ldots \leqslant M_N < \infty.$$

It is easily seen that

$$M_1 M_2\ldots M_N = (m_1 m_2\ldots m_n)^P. \tag{1}$$

Next we associate with each product $M_{\mathrm{K}} = M_{\nu_1\nu_2\ldots\nu_p}$ the point

$$\mathsf{H}^*_{\mathrm{K}} = \mathsf{H}^*_{\nu_1\nu_2\ldots\nu_p} = [X_{\nu_1}, X_{\nu_2},\ldots, X_{\nu_p}] \tag{2}$$

which evidently belongs to Λ. Then $\mathsf{H}^*_1, \mathsf{H}^*_2,\ldots, \mathsf{H}^*_N$ are linearly independent.

First, since $X_1, X_2, \ldots, X_n$ are by hypothesis linearly independent, every point X in R_n is of the form $X = t_1 X_1 + t_2 X_2 + \ldots + t_n X_n$ with real coefficients t_k. Secondly, every point Ξ on the Grassmann manifold $\mathbf{\Omega}(n,p)$ is the compound $\Xi = [X^{(1)}, X^{(2)}, \ldots, X^{(p)}]$ of p suitable points $X^{(1)}, X^{(2)}, \ldots, X^{(p)}$ in R_n, and so can be written in the form

$$\Xi = \tau_1 \mathsf{H}_1^* + \tau_2 \mathsf{H}_2^* + \ldots + \tau_N \mathsf{H}_N^* \tag{3}$$

with real coefficients τ_{K}. Finally, as we saw in § 7, every point in R_N may be expressed as a linear form with real coefficients in finitely many points on $\mathbf{\Omega}(n,p)$ and is thus also an expression (3). But this means that

$$\mathsf{H}_1^*, \mathsf{H}_2^*, \ldots, \mathsf{H}_N^*$$

generate R_N and are therefore linearly independent.

It was proved in § 9 that

$$\Phi(\Xi) = \inf \sum_{\rho=1}^{r} F(X_\rho^{(1)}) F(X_\rho^{(2)}) \ldots F(X_\rho^{(p)}),$$

where the lower bound extends over all finite decompositions

$$\Xi = \sum_{\rho=1}^{r} [X_\rho^{(1)}, X_\rho^{(2)}, \ldots, X_\rho^{(p)}].$$

Hence the special decomposition (2) of $\mathsf{H}_{\mathrm{K}}^*$ gives the inequality

$$\Phi(\mathsf{H}_{\mathrm{K}}^*) \leqslant F(X_{\nu_1}) F(X_{\nu_2}) \ldots F(X_{\nu_p}) = m_{\nu_1} m_{\nu_2} \ldots m_{\nu_p} = M_{\mathrm{K}}. \tag{4}$$

Denote now by $\mathsf{H}_1, \mathsf{H}_2, \ldots, \mathsf{H}_N$ the points $\mathsf{H}_1^*, \mathsf{H}_2^*, \ldots, \mathsf{H}_N^*$ rearranged in such a way that

$$\Phi(\mathsf{H}_1) \leqslant \Phi(\mathsf{H}_2) \leqslant \ldots \leqslant \Phi(\mathsf{H}_N).$$

Then also

$$\Phi(\mathsf{H}_{\mathrm{K}}) \leqslant M_{\mathrm{K}} \qquad (\mathrm{K} = 1, 2, \ldots, N). \tag{5}$$

For the numbers M_{K} were ordered according to increasing size; by (4), none of the first K values $\Phi(\mathsf{H}_1^*), \Phi(\mathsf{H}_2^*), \ldots, \Phi(\mathsf{H}_R^*)$ can then exceed M_{K}.

15. The results desired now follow quickly. On combining the inequalities (5) of the last section with the inequalities (3) in § 12, we find that

$$\mu_{\mathrm{K}} \leqslant \Phi(\mathsf{H}_{\mathrm{K}}) \leqslant M_{\mathrm{K}} \qquad (R = 1, 2, \ldots, N). \tag{1}$$

On the other hand, by the formulae (3) in § 13 and (1) in § 14,

$$\mu_1 \mu_2 \ldots \mu_N \geqslant c_7 (m_1 m_2 \ldots m_n)^P = c_7 M_1 M_2 \ldots M_N,$$

whence

$$\mu_{\mathrm{K}} \geqslant c_7 M_1 M_2 \ldots M_N \prod_{\substack{\mathrm{H}=1 \\ \mathrm{H} \neq \mathrm{K}}}^{N} M_{\mathrm{H}}^{-1} = c_7 M_{\mathrm{K}} \qquad (\mathrm{K} = 1, 2, \ldots, N). \tag{2}$$

The two inequalities (1) and (2) contain the second main result of this paper.

THEOREM 3. *There exists a positive constant c_7 depending only on n and p, as follows.*

Let K be a closed, bounded, symmetric, convex body in R_n, and let $\mathsf{K} = [K]^{(p)}$ be its p-th compound body in R_N; let L be an n-dimensional lattice in R_n, and let Λ be its p-th compound in R_N; and let

$$m_k = m_r(K, L) \qquad (k = 1, 2,..., n)$$

and

$$\mu_{\mathrm{K}} = \mu_{\mathrm{K}}(\mathsf{K}, \Lambda) \qquad (\mathrm{K} = 1, 2,..., N)$$

be the successive minima of K in L, and of K in Λ, respectively. Let the N products

$$M_{\mathrm{K}} = M_{\nu_1\nu_2...\nu_p} = m_{\nu_1} m_{\nu_2}...m_{\nu_p} \quad (1 \leqslant \nu_1 < \nu_2 < ... < \nu_p \leqslant n)$$

be numbered in the order of increasing size. Then

$$c_7 M_{\mathrm{K}} \leqslant \mu_{\mathrm{K}} \leqslant M_{\mathrm{K}} \qquad (\mathrm{K} = 1, 2,..., N).$$

16. In order to connect the last theorem with a known result, we shall study the special case when $p = n-1$, hence $N = n$ and $P = n-1$, a little more in detail.

In this particular case, both K and $\mathsf{K} = [K]^{(n-1)}$ lie in R_n. There is a further convex body in R_n that now becomes of importance, the body denoted by K^{-1} which is polar-reciprocal to K with respect to the unit sphere G_n. This body K^{-1} consists of those points Y in R_n for which

$$|XY| \leqslant 1 \quad \text{for all } X \in K.$$

Here $XY = x_1 y_1 + x_2 y_2 + ... + x_n y_n$ denotes the inner product of the points $X = (x_1, x_2,..., x_n)$ and $Y = (y_1, y_2,..., y_n)$.

Assume, in particular, that K coincides with the unit sphere G_n. The same is then also true for K^{-1} because the hyperplanes $XY = \pm 1$ in Y-space have the distance $1/|X| \geqslant 1$ from O, and so K^{-1} is the intersection of the half-spaces $XY \leqslant 1$ where $|X| = 1$.

Next, the compound body $\mathsf{K} = [K]^{(n-1)}$ now likewise becomes the unit sphere G_n. For the distance function $\Phi(\Xi)$ of K is in this case given by

$$\Phi(\Xi) = \inf \sum_{\rho=1}^{r} |X_\rho^{(1)}||X_\rho^{(2)}|...|X_\rho^{(n-1)}|,$$

where the lower bound extends again over all decompositions

$$\Xi = \sum_{\rho=1}^{r} [X_\rho^{(1)}, X_\rho^{(2)},..., X_\rho^{(n-1)}]$$

of Ξ. Here the compound point $[X^{(1)}, X^{(2)},..., X^{(n-1)}]$ in R_n has as its coordinates the distinct minors of order $n-1$ of the $(n-1) \times n$ matrix formed by the coordinates of the points $X^{(1)}, X^{(2)},..., X^{(n-1)}$. We may assume that these minors have once for all been numbered and given appropriate signs in such a way that the inner product $X.[X^{(1)}, X^{(2)},..., X^{(n-1)}]$ becomes equal to the determinant $\{X, X^{(1)}, X^{(2)}, \dots, X^{(n-1)}\}$, identically in the arbitrary point X. The decomposition of Ξ implies therefore that

$$|\Xi|^2 = \Xi\Xi = \sum_{\rho=1}^{r} \{\Xi, X_\rho^{(1)}, X_\rho^{(2)}, \dots, X_\rho^{(n-1)}\},$$

and here

$$|\{\Xi, X_\rho^{(1)}, X_\rho^{(2)}, \dots, X_\rho^{(n-1)}\}| \leqslant |\Xi|\,|X_\rho^{(1)}|\,|X_\rho^{(2)}| \dots |X_\rho^{(n-1)}|,$$

by Hadamard's determinant theorem. Therefore

$$|\Xi| \leqslant \sum_{\rho=1}^{r} |X_\rho^{(1)}|\,|X_\rho^{(2)}| \dots |X_\rho^{(n-1)}|,$$

whence
$$\Phi(\Xi) \geqslant \sum_{\rho=1}^{r} |X_\rho^{(1)}|\,|X_\rho^{(2)}| \dots |X_\rho^{(n-1)}| \geqslant |\Xi|.$$

Hence K: $\Phi(\Xi) \leqslant 1$ is contained in G_n: $|\Xi| \leqslant 1$. To prove that also $G_n \subseteq \mathsf{K}$, it suffices to show that every point Ξ with $|\Xi| = 1$ belongs to K. We can select $n-1$ points $X^{(1)}, X^{(2)}, \dots, X^{(n-1)}$ on the unit sphere $|X| = 1$ which are orthogonal to Ξ and also in pairs to one another, and for which, moreover, the determinant $\{\Xi, X^{(1)}, X^{(2)}, \dots, X^{(n-1)}\}$ has the value $+1$. The compound point $[X^{(1)}, X^{(2)}, \dots, X^{(n-1)}]$, $= \mathsf{H}$ say, belongs then to K, and it is identical with Ξ because H is likewise orthogonal to all points $X^{(1)}, X^{(2)}, \dots, X^{(n-1)}$ and has the property that

$$\Xi\mathsf{H} = \{\Xi, X^{(1)}, X^{(2)}, \dots, X^{(n-1)}\} = +1.$$

17. There is still a simple connexion between $\mathsf{K} = [K]^{(n-1)}$ and K^{-1} when K is now an *arbitrary* bounded, closed, symmetric, convex body in R_n. Before proving this, let us first consider the effect of an affine transformation $X \to X' = \Omega X$ applied to K on the two corresponding bodies K and K^{-1}. Denote again by $\omega \neq 0$ the determinant of Ω; let further $X, X^{(1)}, X^{(2)}, \dots, X^{(n-1)}$ be n arbitrary points in R_n. From the definition of the compound transformation $\Omega^{(n-1)}$,

$$[\Omega X^{(1)}, \Omega X^{(2)}, \dots, \Omega X^{(n-1)}] = \Omega^{(n-1)}\,[X^{(1)}, X^{(2)}, \dots, X^{(n-1)}].$$

Next, from the multiplication law for determinants,

$$\{\Omega X, \Omega X^{(1)}, \Omega X^{(2)}, \dots, \Omega X^{(n-1)}\} = \omega\,\{X, X^{(1)}, X^{(2)}, \dots, X^{(n-1)}\}.$$

It follows then from the relation between the compound and the determinant given in the last section that

$$\Omega X \,.\, \Omega^{(n-1)}\,[X^{(1)}, X^{(2)}, \dots, X^{(n-1)}] = \omega X \,.\, [X^{(1)}, X^{(2)}, \dots, X^{(n-1)}].$$

In this identity, $[X^{(1)}, X^{(2)}, \dots, X^{(n-1)}]$ can be made to coincide with any given point Y in R_n. Hence

$$\Omega X \,.\, \omega^{-1}\Omega^{(n-1)} Y = X \,.\, Y$$

identically in X and Y.

Therefore, as Ω transforms K into the body ΩK, and as $\mathsf{K} = [K]^{(n-1)}$ simultaneously becomes $\Omega^{(n-1)}\mathsf{K}$, the polar-reciprocal body K^{-1} is at the same time changed into the new body $\omega^{-1}\Omega^{(n-1)}K^{-1}$.

18. The desired connexion between $\mathsf{K} = [K]^{(n-1)}$ and K^{-1} is now easily found. Just as in the proof of Theorem 1, let E be the ellipsoid of smallest volume circumscribed to K so that again

$$n^{-\frac{1}{2}}E \subseteq K \subseteq E. \tag{1}$$

Let further $X \to X' = \Omega X$ denote the affine transformation which changes the unit sphere G_n into $E = \Omega G_n$; it may be assumed, without loss of generality, that the determinant ω of Ω is positive. The compound body $\mathsf{E} = [E]^{(n-1)}$ is then equal to $\mathsf{E} = \Omega^{(n-1)}G_n$ since $[G_n]^{(n-1)} = G_n$. Hence

$$n^{-\frac{1}{2}(n-1)}\Omega^{(n-1)}G_n \subseteq \mathsf{K} \subseteq \Omega^{(n-1)}G_n, \tag{2}$$

in the same way as in the proof of Theorem 1.

An analogous relation holds for K^{-1}. It is obvious from the definition of the polar-reciprocal body that

$$(tK)^{-1} = t^{-1}K^{-1} \text{ for } t > 0, \quad \text{and} \quad K_1^{-1} \supseteq K_2^{-1} \text{ if } K_1 \subseteq K_2.$$

Now $E = \Omega G_n$ and therefore, by the last section,

$$E^{-1} = \omega^{-1}\Omega^{(n-1)}G_n, \qquad (n^{-\frac{1}{2}}E)^{-1} = n^{\frac{1}{2}}\omega^{-1}\Omega^{(n-1)}G_n,$$

because $G_n^{-1} = G_n$. Hence

$$\omega^{-1}\Omega^{(n-1)}G_n \subseteq K^{-1} \subseteq n^{\frac{1}{2}}\omega^{-1}\Omega^{(n-1)}G_n, \tag{3}$$

whence, on combining (2) and (3),

$$n^{-\frac{1}{2}n}\omega K^{-1} \subseteq \mathsf{K} \subseteq \omega K^{-1}. \tag{4}$$

In this inequality, ω has the value

$$\omega = V(E)V(G_n)^{-1},$$

and so, by (1), satisfies the inequality

$$V(K)V(G_n)^{-1} \leqslant \omega \leqslant n^{\frac{1}{2}n}V(K)V(G_n)^{-1}.$$

Finally on substituting these estimates for ω in (4), we find that

$$n^{-\frac{1}{2}n}V(K)V(G_n)^{-1}K^{-1} \subseteq \mathsf{K} \subseteq n^{\frac{1}{2}n}V(K)V(G_n)^{-1}K^{-1},$$

and obtain the following result.

THEOREM 4. *There exist two positive constants c_9 and c_{10} with $c_9 < c_{10}$ depending only on n, with the following property.*

Let K be a closed, bounded, symmetric, convex body in R_n; let K^{-1} be its polar-reciprocal body; and let $\mathsf{K} = [K]^{(n-1)}$ be its $(n-1)$th compound body. Then

$$c_9 V(K)K^{-1} \subseteq \mathsf{K} \subseteq c_{10} V(K)K^{-1}.$$

Remark. One can prove similar relations connecting the bodies $[K^{-1}]^{(p)}$ and $[K]^{(n-p)}$ when $p = 2, 3, \ldots, n-1$.

19. We next introduce the reciprocal of a lattice. Let again L be any n-dimensional lattice in R_n, of basis $Z_1, Z_2, \ldots, Z_n$, say. Then L^{-1} is defined as the set of those points Y in R_n for which

$$XY \text{ is an integer for all } X \in L.$$

One shows without difficulty that L^{-1} is likewise a lattice, viz. the lattice of basis $Z'_1, Z'_2, \ldots, Z'_n$, where these points are defined by the equations

$$Z_h Z'_k = \begin{cases} 1 \text{ if } h = k, \\ 0 \text{ if } h \neq k \end{cases} \qquad (h, k = 1, 2, \ldots, n).$$

It is also easily seen that

$$d(L^{-1}) = \{d(L)\}^{-1}.$$

Again the $(n-1)$th compound lattice $\Lambda = [L]^{(n-1)}$ of L lies in R_n and has the determinant

$$d(\Lambda) = \{d(L)\}^P = \{d(L)\}^{n-1}.$$

We show now that L^{-1} and Λ are similar lattices.

In the special case when L coincides with the lattice L_0 of all points with integral coordinates, it is evident that also $L^{-1} = L_0$ and $\Lambda = L_0$. Let now $X \to X' = \Omega X$ be the affine transformation which changes L_0 into $L = \Omega L_0$; Ω is of determinant $\omega = d(L)$. Then also

$$\Lambda = [\Omega L_0]^{(n-1)} = \Omega^{(n-1)} L_0 \quad \text{and} \quad L^{-1} = \{d(L)\}^{-1} \Omega^{(n-1)} L_0,$$

the second equation following from the formulae given in § 17. Hence

$$L^{-1} = \{d(L)\}^{-1} \Lambda = \{d(L)\}^{-1} [L]^{(n-1)}.$$

20. The following result can now be deduced from Theorems 3 and 4.

THEOREM 5. *There exist two positive constants c_{11} and c_{12} with $c_{11} < c_{12}$ depending only on n, with the following property.*

Let K be a closed, bounded, symmetric, convex body in R_n, and let K^{-1} be its polar-reciprocal body; let L be an n-dimensional lattice, and let L^{-1} be its reciprocal lattice; finally let

$$m_k = m_k(K, L) \quad \textit{and} \quad m'_k = m'_k(K^{-1}, L^{-1}) \quad (k = 1, 2, \ldots, n)$$

be the n successive minima of K in L, and of K^{-1} in L^{-1}, respectively. Then

$$c_{11} \leqslant m'_k m_{n-k+1} \leqslant c_{12} \qquad (k = 1, 2, \ldots, n).$$

Proof. By definition, m'_k is the smallest positive number such that $m'_k K^{-1}$ contains k linearly independent points of L^{-1}; and similarly $\mu_k = \mu_k(\mathsf{K}, \Lambda)$, where $\mathsf{K} = [K]^{(n-1)}$ and $\Lambda = [L]^{(n-1)}$, is the smallest positive number such that $\mu_k \mathsf{K}$ contains k linearly independent points of Λ, and so $\dfrac{\mu_k}{d(L)} \mathsf{K}$ contains k linearly independent points of $\{d(L)\}^{-1}\Lambda = L^{-1}$. Now, by Theorem 4,

$$c_9 \frac{\mu_k}{d(L)} V(K) K^{-1} \subseteq \frac{\mu_k}{d(L)} \mathsf{K} \subseteq c_{10} \frac{\mu_k}{d(L)} V(K) K^{-1}.$$

It follows therefore that

$$c_9 \frac{V(K)}{d(L)} \mu_k \leqslant m'_k \leqslant c_{10} \frac{V(K)}{d(L)} \mu_k \qquad (k = 1, 2, \ldots, n),$$

that is,

$$\frac{1}{c_{10}} \frac{d(L)}{V(K)} m'_k \leqslant \mu_k \leqslant \frac{1}{c_9} \frac{d(L)}{V(K)} m'_k \quad (k = 1, 2, \ldots, n). \quad (1)$$

In the present case $p = n-1$, the numbers M_K of Theorem 3 take the form

$$M_k = \frac{m_1 m_2 \ldots m_n}{m_{n-k+1}} \qquad (k = 1, 2, \ldots, n),$$

because this numbering implies that

$$M_1 \leqslant M_2 \leqslant \ldots \leqslant M_n,$$

as it should be. Theorem 3 states now that

$$c_7 M_k \leqslant \mu_k \leqslant M_k \qquad (k = 1, 2, \ldots, n),$$

so that, in the present case,

$$c_7 m_1 m_2 \ldots m_n \leqslant \mu_k m_{n-k+1} \leqslant m_1 m_2 \ldots m_n \quad (k = 1, 2, \ldots, n).$$

We replace here μ_k by its lower and upper estimates from (1) and obtain the inequalities

$$c_7 c_9 \frac{m_1 m_2 \ldots m_n V(K)}{d(L)} \leqslant m'_k m_{n-k+1} \leqslant c_{10} \frac{m_1 m_2 \ldots m_n V(K)}{d(L)} \qquad (k = 1, 2, \ldots n),$$

where, by Minkowski's theorem (1) in § 12,

$$\frac{2^n}{n!} \leqslant \frac{m_1 m_2 \ldots m_n V(K)}{d(L)} \leqslant 2^n.$$

Therefore, finally,

$$2^n (n!)^{-1} c_7 c_9 \leqslant m'_k m_{n-k+1} \leqslant 2^n c_{10} \qquad (k = 1, 2, \ldots, n),$$

whence the assertion.

Theorem 5 is not new. After an earlier result by M. Riesz (**7**), I proved (**6**) that

$$1 \leqslant m'_k m_{n-k+1} \leqslant (n!)^2 \qquad (k = 1, 2, \ldots, n).$$

Here the upper bound can be further improved by means of recent results in the geometry of numbers, e.g. to

$$m'_k m_{n-k+1} \leqslant \frac{n^{\frac{1}{2}}}{\Delta(G_n)^2},$$

where G_n is the unit sphere in R_n and $\Delta(G_n)$ is its lattice determinant, just as before. In the present paper the detailed proof of Theorem 5 has been given for the sole purpose of showing that this theorem is a consequence of the more general theory of compound bodies.

21. My original work on Theorem 5 arose from the wish to generalize the so-called *transfer principle* of A. Khintchine (**5**) in the theory of Diophantine approximations. We shall now deduce from Theorem 3 a very general transfer principle which contains most of the previous results as special cases.

We choose for the convex body K in Theorem 3 the cube

$$|x_1| \leqslant 1, \quad |x_2| \leqslant 1, \quad ..., \quad |x_n| \leqslant 1$$

of distance function

$$F(X) = \max(|x_1|, |x_2|, ..., |x_n|).$$

The corresponding compound convex body $\mathsf{K} = [K]^{(p)}$ in R_N is somewhat complicated in the case of general p, and it is not quite simple to find its distance function $\Phi(\Xi)$. Fortunately, there is no need to give the exact expression of $\Phi(\Xi)$, and a rather crude inequality will suffice.

Let
$$\Psi(\Xi) = \max(|\xi_1|, |\xi_2|, ..., |\xi_N|)$$

denote the distance function of the cube Q: $\Psi(\Xi) \leqslant 1$ in R_N. There evidently exist two positive constants c_{13} and c_{14} with $c_{13} < c_{14}$ and depending only on n and p such that the cube $c_{14}^{-1}\,Q$: $\Psi(\Xi) \leqslant c_{14}^{-1}$ is contained in K, while K is contained in the cube $c_{13}^{-1}\,Q$: $\Psi(\Xi) \leqslant c_{13}^{-1}$; this follows from O being an inner point of K, and K being bounded. Hence

$$c_{13}\Psi(\Xi) \leqslant \Phi(\Xi) \leqslant c_{14}\Psi(\Xi) \quad \text{for all } \Xi \in R_N, \qquad (1)$$

giving the wanted estimate for $\Phi(\Xi)$.

Let now L be any lattice in R_n of determinant $d(L) = 1$, and let $\Lambda = [L]^{(p)}$ be its pth compound in R_N; then also $d(\Lambda) = 1$. The points $X = (x_1, x_2, ..., x_n)$ of L have the coordinates

$$x_h = \sum_{k=1}^{n} a_{hk}\,u_k \qquad (h = 1, 2, ..., n),$$

where $u_1, u_2, ..., u_n$ run over all integers; the coefficient matrix (a_{hk}) may be assumed to have the determinant $+1$. Similarly, the points

$$\Xi = (\xi_1, \xi_2, ..., \xi_N)$$

of Λ are given by

$$\xi_{\mathrm{H}} = \sum_{\mathrm{K}=1}^{N} a_{\mathrm{HK}}^{(p)}\,v_{\mathrm{K}} \qquad (\mathrm{H} = 1, 2, ..., N),$$

where also $v_1, v_2, ..., v_N$ assume all integral values, and where the coefficient matrix $(a_{\mathrm{HK}}^{(p)})$ is likewise of determinant $+1$, and has as its elements the minors of order p of the original matrix (a_{hk}), arranged in the order that was fixed in § 1.

As before, let $m_k = m_k(K, L)$ and $\mu_{\mathrm{K}} = \mu_{\mathrm{K}}(\mathsf{K}, \Lambda)$ be the successive minima of K in L, and of K in Λ, respectively; also let the products M_{K} be defined as in Theorem 3 so that

$$c_7 M_{\mathrm{K}} \leqslant \mu_{\mathrm{K}} \leqslant M_{\mathrm{K}} \qquad (\mathrm{K} = 1, 2, ..., N). \quad (2)$$

Minkowski's inequality (1) in § 12 takes the form

$$(n!)^{-1} \leqslant m_1 m_2 \dots m_n \leqslant 1, \tag{3}$$

because in the present case $V(K) = 2^n$ and $d(L) = 1$.

Since $m_1 \leqslant m_2 \leqslant \dots \leqslant m_n$, $M_1 = m_1 m_2 \dots m_p$ is the smallest of the N products M_{K}. The minimum value of M_1 is attained when all minima m_k, where $1 \leqslant k \leqslant p$, have the same value, and then $M_1 = m_1^p$. On the other hand, by (3),

$$M_1 \leqslant \frac{1}{m_{p+1} m_{p+2} \dots m_n},$$

and here the right-hand side becomes a maximum when the denominator is a minimum. This is obviously the case when $m_2 = m_3 = \dots = m_n$, and then (3) gives

$$m_2 = m_3 = \dots = m_n \geqslant (n!\, m_1)^{-1/(n-1)},$$

whence

$$M_1 = (m_{p+1} m_{p+2} \dots m_n)^{-1} \leqslant (n!\, m_1)^{(n-p)/(n-1)}.$$

We have thus proved that

$$m_1^p \leqslant M_1 \leqslant (n!\, m_1)^{(n-p)/(n-1)}$$

and therefore, by (2), also

$$c_7 m_1^p \leqslant \mu_1 \leqslant (n!\, m_1)^{(n-p)/(n-1)}.$$

The number μ_1 is the minimum value of $\Phi(\Xi)$ for the points $\Xi \neq O$ of Λ. Now, by (1), the quotient $\Psi(\Xi)/\Phi(\Xi)$ lies between two positive constants that depend only on n and p. Hence, with a slight change of notation, the following theorem has been proved.

THEOREM 6. *There exist two positive constants c_{15} and c_{16} depending only on n and p, with the following property.*

Let (a_{hk}) be a real square matrix of order n and determinant $+1$, and let $(a^{(p)}_{\mathrm{HK}})$ be its p-th compound matrix, which is formed by the minors of order p of the first matrix. Put

$$F(X) = \max_{h=1,2,\dots,n} \left(\left| \sum_{k=1}^{n} a_{hk} x_k \right| \right) \quad \textit{and} \quad \Phi(\Xi) = \max_{\mathrm{H}=1,2,\dots,N} \left(\left| \sum_{\mathrm{K}=1}^{N} a^{(p)}_{\mathrm{HK}} \xi_{\mathrm{K}} \right| \right),$$

and denote by m and μ the minimum of $F(X)$ and that of $\Phi(\Xi)$ at all points $X = (x_1, x_2, \dots, x_n) \neq O$ and $\Xi = (\xi_1, \xi_2, \dots, \xi_N) \neq O$ with integral coordinates, respectively. Then

$$\mu \leqslant c_{15}\, m^{(n-p)/(n-1)} \quad \textit{and} \quad m \leqslant c_{16}\, \mu^{1/p}.$$

Theorem 6 contains most of the older transfer principles as special cases, and it allows similar applications, e.g. to inhomogeneous Diophantine approximations. It is further possible to deduce from it a still more general result involving, in addition to the real linear forms, linear forms with coefficients in one or more p-adic fields.

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