ON COMPOUND CONVEX BODIES (II)

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THE inequality $$0 < c_1 \leqslant V(K)^{-P}\, V(\mathsf{K}) \leqslant c_2$$
for the volume of the pth compound $\mathsf{K} = [K]^{(p)}$ of a single convex body, which was proved in the first part of this paper, cannot be fully extended to the compounds $\mathsf{K} = [K^{(1)}, K^{(2)},\ldots, K^{(p)}]$ of several distinct convex bodies. The problem of estimates for $V(\mathsf{K})$ will be investigated in the present note, and a partial result will be proved.

1. We use the same notation as in the first part. As before, let
$$K^{(1)},\ K^{(2)},\ldots,\ K^{(p)}$$
be any p closed, bounded, symmetric, convex bodies in R_n, and let
$$\mathsf{K} = [K^{(1)}, K^{(2)},\ldots, K^{(p)}]$$
be their compound in R_N. We wish either to find upper and lower bounds for the volume $V(\mathsf{K})$ in terms of some symmetric function of
$$V(K^{(1)}),\ V(K^{(2)}),\ldots,\ V(K^{(p)}),$$
or to show that such bounds do not exist. Since, for positive $t_1, t_2,\ldots, t_p$,
$$[t_1 K^{(1)}, t_2 K^{(2)},\ldots, t_p K^{(p)}] = t_1 t_2 \ldots t_p\, \mathsf{K},$$
and since further
$$V(t_1 t_2 \ldots t_p\, \mathsf{K}) = (t_1 t_2 \ldots t_p)^N\, V(\mathsf{K}),$$
$$V(t_1 K^{(1)}) = t_1^n\, V(K^{(1)}), \quad \ldots, \quad V(t_p K^{(p)}) = t_p^n\, V(K^{(p)}),$$
it is clear, for reasons of homogeneity, that we must compare $V(\mathsf{K})$ with the expression
$$\left\{\prod_{\pi=1}^{p} V(K^{(\pi)})\right\}^{P/p}.$$
The question is therefore whether
$$S(\mathsf{K}) = V(\mathsf{K})\left\{\prod_{\pi=1}^{p} V(K^{(\pi)})\right\}^{-P/p}$$
possesses positive upper and lower bounds depending only on n and p.

2. For the upper bound, the problem is solved by the following theorem.

THEOREM 1. *Let $n \geqslant 3$ and $2 \leqslant p \leqslant n-1$, and let $c > 0$ be arbitrary. Then there exist p closed, bounded, symmetric, convex bodies $K^{(1)}, K^{(2)},\ldots, K^{(p)}$ such that their compound $\mathsf{K} = [K^{(1)}, K^{(2)},\ldots, K^{(p)}]$ satisfies the inequality*
$$S(\mathsf{K}) > c.$$
Thus $S(\mathsf{K})$ admits of no upper bound depending only on n and p.

Proof. We choose for $K^{(1)} = K^{(2)} = \ldots = K^{(p-1)}$ the generalized octahedron

$$|x_1|+|x_2|+\ldots+|x_n| \leqslant 1,$$

and for $K^{(p)}$ the generalized octahedron

$$\frac{1}{a}\{|x_1|+|x_2|+\ldots+|x_{n-1}|\}+a^{n-1}|x_n| \leqslant 1,$$

where a is a parameter satisfying $0 < a \leqslant 1$. Then

$$V(K^{(1)}) = V(K^{(2)}) = \ldots = V(K^{(p)}) = \frac{2^n}{n!}.$$

The first $p-1$ octahedra have the vertices

$$\pm U_1, \pm U_2, \ldots, \pm U_n,$$

where

$$U_1 = (1, 0, \ldots, 0), \quad U_2 = (0, 1, \ldots, 0), \quad \ldots, \quad U_n = (0, 0, \ldots, 1)$$

denote the n unit points on the coordinate axes in R_n. Similarly the vertices of the last octahedron lie at

$$\pm aU_1, \pm aU_2, \ldots, \pm aU_{n-1}, \pm a^{-(n-1)}U_n.$$

The compound body $\mathsf{K} = [K^{(1)}, K^{(2)}, \ldots, K^{(p)}]$ contains, in particular, the convex hull H of the $2N$ compound points

$$\pm a_{\nu_p}[U_{\nu_1}, U_{\nu_2}, \ldots, U_{\nu_p}]. \tag{1}$$

Here $\nu_1, \nu_2, \ldots, \nu_p$ run over all N distinct sets of p indices satisfying

$$1 \leqslant \nu_1 < \nu_2 < \ldots < \nu_p \leqslant n,$$

and we have, for shortness, put

$$a_{\nu_p} = a \text{ if } \nu_p = 1, 2, \ldots, n-1, \quad \text{but} \quad a_{\nu_p} = a^{-(n-1)} \text{ if } \nu_p = n.$$

The N compounds $[U_{\nu_1}, U_{\nu_2}, \ldots, U_{\nu_p}]$ coincide with the unit points on the N coordinate axes in R_N; evidently exactly P of them belong to $\nu_p = n$. Hence all points (1) lie on the coordinate axes; just $2P$ of them have one coordinate equal to $\pm a^{-(n-1)}$ and the other coordinates equal to zero; and each of the remaining $2(N-P)$ points has just one coordinate equal to $\pm a$ and the other coordinates equal to zero. Thus, if the numbering of the coordinates $\xi_1, \xi_2, \ldots, \xi_N$ of the general point Ξ in R_N is chosen suitably, then the convex hull H of the points (1) becomes the generalized octahedron

$$\frac{1}{a}\sum_{k=1}^{N-P}|\xi_k|+a^{n-1}\sum_{k=N-P+1}^{N}|\xi_k| \leqslant 1$$

of volume

$$V(\mathsf{H}) = \frac{2^N}{N!}a^{(N-P)-(n-1)P}.$$

Since H is a subset of K, this implies that

$$V(\mathsf{K}) \geqslant \frac{2^N}{N!}a^{(N-P)-(n-1)P},$$

and we therefore obtain the inequality

$$S(\mathsf{K}) = V(\mathsf{K})\Big\{\prod_{\pi=1}^{p} V(K^{(\pi)})\Big\}^{-P/p} \geqslant \frac{2^N}{N!}\binom{2n}{n!}^{-P} a^{(N-P)-(n-1)P}.$$

Here the expression on the right-hand side can be made arbitrarily large by choosing a sufficiently small because

$$(N-P)-(n-1)P = N-nP = \binom{n}{p}-n\binom{n-1}{p-1} = -\frac{n(p-1)}{p}\binom{n-1}{p-1} < 0.$$

This proves the assertion.

3. It is much more difficult to decide whether $S(\mathsf{K})$ possesses any positive lower bound that depends only on n and p. In this note the problem will be settled in the special case when $n \geqslant 3$, $2 \leqslant p \leqslant n-1$, and when $K^{(1)}, K^{(2)},..., K^{(p)}$ are made up by repetition of just *two* distinct convex bodies.

To fix the notation, let $p = r+s$, $r \geqslant 1$, $s \geqslant 1$; assume that the first r of the bodies $K^{(1)}, K^{(2)},..., K^{(p)}$ are identical with K_1, and that the last s bodies are identical with K_2. We then write, for shortness,

$$\mathsf{K} = [K_1^r\, K_2^s],$$

and the number $S(\mathsf{K})$ takes the form

$$S(\mathsf{K}) = V(\mathsf{K})\,\{V(K_1)^r V(K_2)^s\}^{-P/p}.$$

We have to show that $S(\mathsf{K})$ is not smaller than a certain positive number which depends only on n and p.

4. Let us begin with the simpler case when $K_1 = E_1$ and $K_2 = E_2$ are ellipsoids in R_N with centres at the origin. By the theory of reduction to principal axes for such ellipsoids, there exists a non-singular affine transformation $X \rightarrow X' = \Omega X$ of R_n such that

$$E_1 = \Omega G_n, \qquad E_2 = \Omega E.$$

Here G_n denotes the unit sphere

$$x_1^2+x_2^2+...+x_n^2 \leqslant 1,$$

and E is an ellipsoid of the form

$$\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\cdots+\frac{x_n^2}{a_n^2} \leqslant 1,$$

the semi-axes of which may be assumed to satisfy the inequalities

$$0 < a_1 \leqslant a_2 \leqslant ... \leqslant a_n. \tag{2}$$

Evidently $$\mathsf{K} = [E_1^r\, E_2^s] = \Omega^{(p)}[G_n^r\, E^s],$$

where $\Omega^{(p)}$ is the pth compound of Ω. Denote by ω the determinant of Ω; then

$$\omega^{(p)} = \omega^P$$

is the determinant of $\Omega^{(p)}$. Let further

$$\mathsf{K}_0 = [G_n^r\, E^s];$$

hence
$$\mathsf{K} = \Omega^{(p)}\mathsf{K}_0.$$

The volumes of G_n and E are given by

$$V(G_n) = \kappa_n, \qquad V(E) = \kappa_n\, a_1\, a_2 \ldots a_n,$$

where κ_n is the constant $\qquad \kappa_n = \dfrac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+1)}.$

Therefore also

$$V(E_1) = \kappa_n |\omega|, \qquad V(E_2) = \kappa_n\, a_1\, a_2 \ldots a_n |\omega|.$$

On the other hand,

$$V(\mathsf{K}) = |\omega^{(p)}| V(\mathsf{K}_0) = |\omega|^P V(\mathsf{K}_0).$$

Hence $\qquad S(\mathsf{K}) = |\omega|^P V(\mathsf{K}_0)\{(\kappa_n|\omega|)^r(\kappa_n\, a_1\, a_2 \ldots a_n |\omega|)^s\}^{-P/p},$

and this may be simplified to

$$S(\mathsf{K}) = \frac{V(\mathsf{K}_0)}{\kappa_n^P(a_1\, a_2 \ldots a_n)^{sP/p}}. \tag{3}$$

Denote again by $\nu_1, \nu_2, \ldots, \nu_p$ all N sets of p indices satisfying

$$1 \leqslant \nu_1 < \nu_2 < \ldots < \nu_p \leqslant n,$$

and, for each such set, put

$$A(\nu) = a_{\nu_1} a_{\nu_2} \ldots a_{\nu_r}, \qquad B(\nu) = a_{\nu_{r+1}} a_{\nu_{r+2}} \ldots a_{\nu_{r+s}}.$$

The product $\qquad \prod_\nu \big(A(\nu)B(\nu)\big) = \prod_\nu (a_{\nu_1} a_{\nu_2} \ldots a_{\nu_p})$

extended over all N sets is easily seen to be equal to

$$(a_1\, a_2 \ldots a_n)^P.$$

On the other hand, the hypothesis (2) gives the inequalities

$$B(\nu) \geqslant \big(A(\nu)B(\nu)\big)^{s/p},$$

and it follows therefore that

$$\prod_\nu B(\nu) \geqslant (a_1\, a_2 \ldots a_n)^{sP/p}. \tag{4}$$

We can now derive a lower bound for $V(\mathsf{K}_0)$; it would be much harder to determine the exact value of this number.

The unit sphere G_n contains the $2n$ positive and negative unit points

$$\pm U_1, \pm U_2, \ldots, \pm U_n,$$

and the ellipsoid E contains the proportional points

$$\pm a_1\, U_1, \pm a_2\, U_2, \ldots, \pm a_n\, U_n.$$

Hence $\mathsf{K}_0 = [G_n^r\, E^s]$ contains all the compound points

$$\pm B(\nu)\,[U_{\nu_1}, U_{\nu_2}, \ldots, U_{\nu_p}]. \tag{5}$$

Apart from the numerical factors $\pm B(\nu)$, these points are just all the N distinct unit points on the coordinate axes in R_N. Hence the convex hull H of the points (5) is a generalized octahedron of volume

$$V(\mathsf{H}) = \frac{2^N}{N!} \prod_\nu B(\nu).$$

Since $\mathsf{K}_0 \supseteq \mathsf{H}$, it follows then by (4) that

$$V(\mathsf{K}_0) \geqslant \frac{2^N}{N!} \prod_\nu B(\nu) \geqslant \frac{2^N}{N!} (a_1 a_2 \ldots a_n)^{sP/p}.$$

We finally substitute this lower bound for $V(\mathsf{K}_0)$ in (3) and obtain the estimate

$$S(\mathsf{K}) \geqslant \frac{2^N}{\kappa_n^P N!}.$$

As asserted, the constant on the right-hand side depends only on n and p.

5. The already asserted result can now be proved.

THEOREM 2. *Let* $n \geqslant 3$, $2 \leqslant p \leqslant n-1$, $p = r+s$, $r \geqslant 1$, $s \geqslant 1$. *Let further* K_1 *and* K_2 *be two closed, bounded, symmetric, convex bodies in* R_n, *and let* $\mathsf{K} = [K_1^r K_2^s]$ *be a mixed compound of these bodies in* R_N. *A positive constant c depending only on n and p exists such that*

$$V(\mathsf{K}) \geqslant c\{V(K_1)^r V(K_2)^s\}^{P/p}.$$

Proof. By the theorem of John (**1**) there exist two ellipsoids E_1 and E_2 in R_n with their centres at the origin such that

$$n^{-\frac{1}{2}} E_1 \subseteq K_1 \subseteq E_1, \qquad n^{-\frac{1}{2}} E_2 \subseteq K_2 \subseteq E_2.$$

Hence, if K_1 is the compound body

$$\mathsf{K}_1 = [E_1^r E_2^s],$$

then

$$n^{-\frac{1}{2}p} \mathsf{K}_1 \subseteq \mathsf{K} \subseteq \mathsf{K}_1,$$

and so also

$$n^{-\frac{1}{2}pN} V(\mathsf{K}_1) \leqslant V(\mathsf{K}) \leqslant V(\mathsf{K}_1). \tag{6}$$

It has already been proved that

$$V(\mathsf{K}_1) \geqslant \frac{2^N}{\kappa_n^P N!} \{V(E_1)^r V(E_2)^s\}^{P/p}.$$

Hence it follows from the left-hand inequality in (6) that

$$V(\mathsf{K}) \geqslant \frac{2^N}{\kappa_n^P N!\, n^{\frac{1}{2}pN}} \{V(E_1)^r V(E_2)^s\}^{P/p} \geqslant \frac{2^N}{\kappa_n^P N!\, n^{\frac{1}{2}pN}} \{V(K_1)^r V(K_2)^s\}^{P/p},$$

as was to be proved.

REFERENCE

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