A PROPERTY OF THE STAR DOMAIN $|xy| \leqslant 1$

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It is well known that the star domain

$$K: \quad |xy| \leqslant 1$$

in the plane is boundedly reducible, that is, it contains a bounded star body with the same lattice determinant, namely $\sqrt{5}$. Hence the bounded star domain

$$K_r: \quad |xy| \leqslant 1, \quad x^2+y^2 \leqslant r^2$$

has the same lattice determinant as K has if r is sufficiently large. The following result is therefore perhaps a little surprising.

Theorem. *Let P be any star polygon contained in K which is bounded by finitely many line segments. Then $\Delta(P) < \Delta(K)$.*

Proof. The boundary of P meets the hyperbolae $xy = \pm 1$ in at most finitely many points, $\pm X_1, \ldots, \pm X_k$ say. Let Λ_0 be any critical lattice of K, and let $t > 0$ be arbitrary. The linear mapping

$$T: \quad x \rightarrow tx, \ y \rightarrow t^{-1} y$$

of unit determinant is an automorphism of K, hence changes Λ_0 into a new critical lattice $\Lambda_t = T\Lambda_0$ of K. In any bounded region there can be at most finitely many points of Λ_t. Hence t may be chosen in such a way that Λ_t contains none of the points $\pm X_1, \ldots, \pm X_k$. This implies that the points of Λ_t distinct from the origin have a positive minimum distance from P. Thus there must exist a neighbouring lattice Λ^* which is P-admissible, but is of smaller determinant than

$$d(\Lambda_t) = d(\Lambda_0) = \Delta(K),$$

whence the assertion.

A similar result holds for polyhedra inscribed in the three-dimensional star body $|xyz| \leqslant 1$. It would be of interest to give analogous examples for more than three dimensions, and to decide generally which star bodies contain polyhedra of equal determinant.

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