large alum crystal, the others were made of wood, glass or brass. The agreement between theory and experiment is as good as could be expected.

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# A FACTORIAL SERIES FOR THE RATIONAL MULTIPLES OF $e$

By K. Mahler

A special case of a theorem by G. Cantor* states that every real number $\alpha$ can be written in a unique way as a series

$$\alpha = \sum_{n=1}^{\infty} \frac{g_n}{n!} \qquad \ldots\ldots(1)$$

where the coefficients $g_n$ are integers, $g_1$ being arbitrary, while

$$0 \leqslant g_n \leqslant n-1 \quad \text{for all } n \geqslant 2 \qquad \ldots\ldots(2)$$

and

$$0 \leqslant g_n \leqslant n-2 \quad \text{for infinitely many } n \geqslant 2. \qquad \ldots\ldots(3)$$

One finds, in fact, that

$$g_1 = [\alpha], \quad \text{and} \quad g_n = [n!\,\alpha] - n[(n-1)!\,\alpha] \quad \text{for } n \geqslant 2,$$

and that, more precisely,

$$\alpha = \sum_{n=1}^{N} \frac{g_n}{n!} + \frac{\alpha_N}{N!}$$

where

$$\alpha_N = N!\,\alpha - [N!\,\alpha] = N! \sum_{n=N+1}^{\infty} \frac{g_n}{n!}, \quad 0 \leqslant \alpha_N < 1.$$

Our aim is to construct the series (1) in the special case when $\alpha$ is a rational multiple of $e$. For simplicity we shall, however, assume that

$$\alpha = \frac{p}{q}\,e, \text{ where } p \text{ and } q \text{ are integers, and } 1 \leqslant p \leqslant q-1. \qquad \ldots\ldots(4)$$

The developments of other rational multiples of $e$ may be obtained by adding suitable integral multiples of one of the series

$$e = 2 + \sum_{n=2}^{\infty} \frac{1}{n!}, \quad -e = -3 + \sum_{n=3}^{\infty} \frac{n-2}{n!}.$$

1. The classical series

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

may be written as

$$e = \sum_{m=0}^{\infty} \sum_{k=0}^{q-1} \frac{1}{(mq+k)!}.$$

We therefore shall try to find integers $a_k$, $b_k$ such that

$$\frac{p}{q} \sum_{k=0}^{q-1} \frac{1}{(mq+k)!} = \sum_{k=0}^{q-1} \frac{a_k m + b_k}{(mq+k+1)!} \qquad \ldots\ldots(5)$$

identically in $m$. For this identity implies that

$$\frac{p}{q}e = \sum_{m=0}^{\infty}\sum_{k=0}^{q-1}\frac{a_k m + b_k}{(mq+k+1)!}, \qquad \ldots\ldots(5a)$$

giving the required series, provided that

$$0 \leqslant a_k m + b_k \leqslant mq + k \qquad \ldots\ldots(6)$$

for all pairs of integers $k$, $m$ with $mq + k \geqslant 1$, and

$$0 \leqslant a_k m + b_k \leqslant mq + k - 1 \qquad \ldots\ldots(7)$$

for infinitely many such pairs.

2. The identity (5) is equivalent to

$$\sum_{k=0}^{q-1}\frac{p}{(mq+k)!} = \sum_{k=0}^{q-1}\frac{(a_k m + b_k)q}{(mq+k+1)!} = \sum_{k=0}^{q-1}\left\{\frac{a_k}{(mq+k)!} + \frac{b_k q - (k+1)a_k}{(mq+k+1)!}\right\}$$

$$= \frac{a_0}{(mq)!} + \sum_{k=1}^{q-1}\frac{a_k + b_{k-1}q - ka_{k-1}}{(mq+k)!} + \frac{(b_{q-1} - a_{q-1})q}{(mq+q)!}.$$

It is therefore satisfied if

$$a_0 = p,$$
$$a_k + b_{k-1}q - ka_{k-1} = p \qquad (k = 1, 2, \ldots, q-1),$$
$$b_{q-1} = a_{q-1}.$$

It thus suffices to choose

$$a_k = \begin{cases} p & \text{if } k = 0, \\ p + ka_{k-1} - b_{k-1}q = (p + ka_{k-1}) - \left[\dfrac{p + ka_{k-1}}{q}\right]q & \text{if } k = 1, 2, \ldots, q-1 \end{cases} \qquad \ldots\ldots(8)$$

and

$$b_k = \begin{cases} \left[\dfrac{p + (k+1)a_k}{q}\right] & \text{if } k = 0, 1, \ldots, q-2, \\ a_{q-1} & \text{if } k = q-1. \end{cases} \qquad \ldots\ldots(9)$$

3. Since $1 \leqslant p \leqslant q - 1$, evidently

$$0 \leqslant a_k \leqslant q - 1 \qquad (k = 0, 1, \ldots, q-1). \qquad \ldots\ldots(10)$$

Further

$$0 \leqslant b_k \leqslant k + 1 \qquad (k = 0, 1, \ldots, q-1). \qquad \ldots\ldots(11)$$

For $b_{q-1} = a_{q-1}$, and so this inequality holds for $k = q - 1$; if, however, $k = 0, 1, \ldots, q - 2$, then

$$0 \leqslant b_k \leqslant \frac{p + (k+1)a_k}{q} \leqslant \frac{(q-1) + (k+1)(q-1)}{q} < k + 2,$$

hence $\leqslant k + 1$.

From (10) and (11),

$$0 \leqslant a_k m + b_k \leqslant (q-1)m + (k+1) = (qm + k) - (m - 1).$$

Hence the condition (6) is certainly satisfied when $m \geqslant 1$ and the condition (7) when $m \geqslant 2$. It follows that all but the $q$ terms

$$\sum_{k=0}^{q-1}\frac{b_k}{(k+1)!} \qquad \ldots\ldots(12)$$

of the series $(A)$ corresponding to $m = 0$, $k = 0, 1, \ldots, q-1$ have the required form, and this series gives the development (1) for $(p/q)e$ except perhaps for its first $q$ terms. We have thus the following result.

THEOREM 1: *Let* $1 \leqslant p \leqslant q-1$. *In the development* (1) *for* $(p/q)e$ *all but the first* $q$ *coefficients* $g_n$ *have the explicit form*

$$g_n = a_k m + b_k \quad \text{if} \quad n = mq + k + 1, k = 0, 1, \ldots, q-1, m \geqslant 1, \qquad \ldots\ldots(13)$$

*where* $a_k$ *and* $b_k$ *are defined by the recursive formulae* (8) *and* (9).

In other words, all but finitely many of the coefficients $g_n$ form $q$ separate arithmetic progressions when $n$ runs over the different residue classes (mod $q$).

4. In addition to the recursive formulae (8) and (9), there are also explicit expressions for $a_k$ and $b_k$.

Put

$$c_k = k!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{k!}\right) \quad (k = 0, 1, 2, \ldots), \qquad \ldots\ldots(14)$$

so that $c_k$ is a positive integer, and

$$c_0 = 1, \qquad c_k = 1 + kc_{k-1} \quad \text{if} \quad k \geqslant 1. \qquad \ldots\ldots(15)$$

Then, by (8), the expression

$$d_k = a_k - pc_k$$

satisfies the congruence

$$d_k \equiv (p + ka_{k-1}) - p(1 + kc_{k-1}) \equiv kd_{k-1} \pmod q.$$

Since evidently $d_0 = 0$, this implies for all $k \geqslant 0$ that $d_k \equiv 0 \pmod q$ and therefore that

$$a_k \equiv pc_k \pmod q.$$

But then, by (10), necessarily

$$a_k = pc_k - \left[\frac{p}{q} c_k\right] q \qquad \ldots\ldots(16)$$

for all values of $k \geqslant 0$.

Next, on substituting this expression for $a_k$ in (9), we find that

$$b_k = \left[\frac{p}{q} + \frac{k+1}{q} pc_k - (k+1)\left[\frac{p}{q} c_k\right]\right]$$

and hence that

$$b_k = \left[\frac{p}{q}(1 + (k+1)c_k)\right] - (k+1)\left[\frac{p}{q} c_k\right] \qquad \ldots\ldots(17)$$

for all $k \geqslant 0$, including the case when $k = q-1$ because then the right-hand side is equal to

$$\left[\frac{p}{q}\right] + pc_{q-1} - q\left[\frac{p}{q} c_{q-1}\right] = a_{q-1}, \qquad \text{since} \left[\frac{p}{q}\right] = 0.$$

The integers $c_k$ increase rapidly. Therefore it proves to be preferable to use the recursive formulae (8) and (9) rather than the explicit expressions (16) and (17) for the actual computation of $a_k$ and $b_k$. It may have some interest to study the arithmetical properties of these coefficients.

5. The following two tables give, (i) the lowest cases of the series $(A)$, and (ii) a table of the coefficients $a_k$ and $b_k$.

*Table of series:*

$$\frac{e}{2} = \frac{1}{1!} + \sum_{m=1}^{\infty} \frac{m+1}{(2m+1)!},$$

$$\frac{e}{3} = \frac{1}{2!} + \frac{2}{3!} + \sum_{m=1}^{\infty} \left(\frac{m}{(3m+1)!} + \frac{2m+1}{(3m+2)!} + \frac{2m+2}{(3m+3)!}\right),$$

$$\frac{2e}{3} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \sum_{m=1}^{\infty} \left(\frac{2m+1}{(3m+1)!} + \frac{m+1}{(3m+2)!} + \frac{m+1}{(3m+3)!}\right),$$

$$\frac{e}{4} = \frac{1}{2!} + \frac{1}{3!} + \sum_{m=1}^{\infty} \left(\frac{m}{(4m+1)!} + \frac{2m+1}{(4m+2)!} + \frac{m+1}{(4m+3)!}\right),$$

$$\frac{3e}{4} = \frac{2}{1!} + \sum_{m=1}^{\infty} \left(\frac{3m+1}{(4m+1)!} + \frac{2m+1}{(4m+2)!} + \frac{3m+3}{(4m+3)!}\right),$$

$$\frac{e}{5} = \frac{1}{2!} + \frac{1}{4!} + \sum_{m=1}^{\infty} \left(\frac{m}{(5m+1)!} + \frac{2m+1}{(5m+2)!} + \frac{m+1}{(5m+4)!}\right),$$

$$\frac{2e}{5} = \frac{1}{1!} + \frac{2}{4!} + \sum_{m=1}^{\infty} \left(\frac{2m}{(5m+1)!} + \frac{4m+2}{(5m+2)!} + \frac{2m+2}{(5m+4)!}\right),$$

$$\frac{3e}{5} = \frac{1}{1!} + \frac{1}{2!} + \frac{3}{4!} + \sum_{m=1}^{\infty} \left(\frac{3m+1}{(5m+1)!} + \frac{m+1}{(5m+2)!} + \frac{3m+3}{(5m+4)!}\right),$$

$$\frac{4e}{5} = \frac{2}{1!} + \frac{1}{3!} + \sum_{m=1}^{\infty} \left(\frac{4m+1}{(5m+1)!} + \frac{3m+2}{(5m+2)!} + \frac{4m+4}{(5m+4)!}\right).$$

*Table of coefficients:*

| $\frac{p}{q}=$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ | $\frac{1}{6}$ | $\frac{5}{6}$ | $\frac{1}{7}$ | $\frac{2}{7}$ | $\frac{3}{7}$ | $\frac{4}{7}$ | $\frac{5}{7}$ | $\frac{6}{7}$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{5}{8}$ | $\frac{7}{8}$ | $\frac{1}{9}$ | $\frac{2}{9}$ | $\frac{4}{9}$ | $\frac{5}{9}$ | $\frac{7}{9}$ | $\frac{8}{9}$ | $\frac{1}{10}$ | $\frac{3}{10}$ | $\frac{7}{10}$ | $\frac{9}{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a_0$ | 1 | 1 | 2 | 1 | 3 | 1 | 2 | 3 | 4 | 1 | 5 | 1 | 2 | 3 | 4 | 5 | 6 | 1 | 3 | 5 | 7 | 1 | 2 | 4 | 5 | 7 | 8 | 1 | 3 | 7 | 9 |
| $b_0$ | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| $a_1$ | 0 | 2 | 1 | 2 | 2 | 2 | 4 | 1 | 3 | 2 | 4 | 2 | 4 | 6 | 1 | 3 | 5 | 2 | 6 | 2 | 6 | 2 | 4 | 8 | 1 | 5 | 7 | 2 | 6 | 4 | 8 |
| $b_1$ | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 0 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 1 | 2 |
| $a_2$ | | 2 | 1 | 1 | 3 | 0 | 0 | 0 | 0 | 5 | 1 | 5 | 3 | 1 | 6 | 4 | 2 | 5 | 7 | 1 | 3 | 5 | 1 | 2 | 7 | 8 | 4 | 5 | 5 | 5 | 5 |
| $b_2$ | | 2 | 1 | 1 | 3 | 0 | 0 | 0 | 0 | 2 | 1 | 2 | 1 | 0 | 3 | 2 | 1 | 2 | 3 | 1 | 2 | 1 | 0 | 1 | 2 | 3 | 2 | 1 | 1 | 2 | 2 |
| $a_3$ | | | | 0 | 0 | 1 | 2 | 3 | 4 | 4 | 2 | 2 | 4 | 6 | 1 | 3 | 5 | 0 | 0 | 0 | 0 | 7 | 5 | 1 | 8 | 4 | 2 | 6 | 8 | 2 | 4 |
| $b_3$ | | | | 0 | 0 | 1 | 2 | 3 | 4 | 2 | 2 | 1 | 2 | 3 | 1 | 2 | 3 | 0 | 0 | 0 | 0 | 3 | 2 | 0 | 4 | 2 | 1 | 2 | 3 | 1 | 2 |
| $a_4$ | | | | | | 0 | 0 | 0 | 0 | 5 | 1 | 2 | 4 | 6 | 1 | 3 | 5 | 1 | 3 | 5 | 7 | 2 | 4 | 8 | 1 | 5 | 7 | 5 | 5 | 5 | 5 |
| $b_4$ | | | | | | 0 | 0 | 0 | 0 | 4 | 1 | 1 | 3 | 4 | 1 | 2 | 4 | 0 | 2 | 3 | 5 | 1 | 2 | 4 | 1 | 3 | 4 | 2 | 2 | 3 | 3 |
| $a_5$ | | | | | | | | | | 2 | 4 | 4 | 1 | 5 | 2 | 6 | 3 | 6 | 2 | 6 | 2 | 2 | 4 | 8 | 1 | 5 | 7 | 6 | 8 | 2 | 4 |
| $b_5$ | | | | | | | | | | 2 | 4 | 3 | 1 | 4 | 2 | 5 | 3 | 4 | 1 | 5 | 2 | 1 | 2 | 5 | 1 | 4 | 5 | 3 | 5 | 1 | 3 |
| $a_6$ | | | | | | | | | | | | 4 | 1 | 5 | 2 | 6 | 3 | 5 | 7 | 1 | 3 | 4 | 8 | 7 | 2 | 1 | 5 | 7 | 1 | 9 | 3 |
| $b_6$ | | | | | | | | | | | | 4 | 1 | 5 | 2 | 6 | 3 | 4 | 6 | 1 | 3 | 3 | 6 | 5 | 2 | 1 | 4 | 5 | 1 | 7 | 3 |
| $a_7$ | | | | | | | | | | | | | | | | | | 4 | 4 | 4 | 4 | 2 | 4 | 8 | 1 | 5 | 7 | 0 | 0 | 0 | 0 |
| $b_7$ | | | | | | | | | | | | | | | | | | 4 | 4 | 4 | 4 | 1 | 3 | 7 | 1 | 5 | 7 | 0 | 0 | 0 | 0 |
| $a_8$ | | | | | | | | | | | | | | | | | | | | | | 8 | 7 | 5 | 4 | 2 | 1 | 1 | 3 | 7 | 9 |
| $b_8$ | | | | | | | | | | | | | | | | | | | | | | 8 | 7 | 5 | 4 | 2 | 1 | 1 | 3 | 7 | 9 |
| $a_9$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | 0 | 0 | 0 | 0 |
| $b_9$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | 0 | 0 | 0 | 0 |

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**1905.** Few people, I think, realized that (Belloc) was a considerable mathematician, but you were aware of it when you heard him talk about the technical details of bridges or about squaring the circle.—J. B. Morton, *Hilaire Belloc: a memoir*, (Hollis and Carter, 1955), p. 39. [Per Professor T. A. A. Broadbent.]