On the zeros of the derivative of a polynomial

By K. Mahler, F.R.S.

Mathematics Department, University of Manchester

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A general property of arbitrary polynomials is proved, one form of which is as follows. If $f(z)$ has the zeros $\alpha_1, \ldots, \alpha_m$, and $f'(z)$ has the zeros $\beta_1, \ldots, \beta_{m-1}$, then

$$\prod_{j=1}^{m-1} \max(1, |\beta_j|) \leqslant \prod_{j=1}^{m} \max(1, |\alpha_j|).$$

1. If
$$f(z) = a_0 z^m + a_1 z^{m-1} + \ldots + a_m$$
is an arbitrary polynomial with real or complex coefficients, we write
$$L(f) = |a_0| + |a_1| + \ldots + |a_m|$$
and call $L(f)$ the *length* of $f(z)$. We further put
$$M(f) = 0 \quad \text{if} \quad f(z) \text{ vanishes identically}$$
and otherwise define a positive number $M(f)$ by
$$M(f) = \exp\left(\int_0^1 \log |f(e^{2\pi i\vartheta})|\, d\vartheta\right);$$
$M(f)$ will be called the *measure* of $f(z)$.

When $f(z)$ reduces to a constant c, $M(f)$ becomes the absolute value $|c|$ of this constant. If further $g(z)$ is a second polynomial,
$$M(fg) = M(f)\, M(g). \tag{1}$$
By means of Jensen's formula I have shown elsewhere (Mahler 1960) that if $f(z)$ has the exact degree m, the highest coefficient $a_0 \neq 0$, and the zeros $\alpha_1, \ldots, \alpha_m$ where, as always in this paper, each zero is written as often as its multiplicity, then
$$M(f) = |a_0| \prod_{j=1}^{m} \max(1, |\alpha_j|) \tag{2}$$
and
$$2^{-m} L(f) \leqslant M(f) \leqslant L(f). \tag{3}$$

The aim of this paper is to prove a general property that connects the zeros of $f(z)$ with those of its derivative $f'(z)$. This property can be expressed in several equivalent forms which we now state as theorems.

Theorem 1. *If $f(z)$ has the exact degree m, then*
$$M(f') \leqslant m M(f).$$

Theorem 2. *If $f(z)$ has the zeros $\alpha_1, \ldots, \alpha_m$, and $f'(z)$ has the zeros $\beta_1, \ldots, \beta_{m-1}$, and and if ρ is any positive number, then*
$$\rho \prod_{j=1}^{m-1} \max(\rho, |\beta_j|) \leqslant \prod_{j=1}^{m} \max(\rho, |\alpha_j|).$$

THEOREM 3. *If* $\alpha_1, \ldots, \alpha_m$ *are arbitrary complex numbers, then*

$$\int_0^1 \log \left| \sum_{j=1}^{m} \frac{1}{e^{2\pi i\vartheta} - \alpha_j} \right| d\vartheta \leqslant \log m.$$

It is easy to see that each of these three theorems implies the other two. First, if the highest coefficient of $f(z)$ is a_0, that of $f'(z)$ is ma_0 where m is the degree of $f(z)$. Hence, by (2), theorem 1 and the case $\rho = 1$ of theorem 2 are equivalent. The general case of theorem 2 is now obtained on applying this special result to the polynomial $f(\rho z)$.

Secondly, the factorization

$$f(z) = a_0(z-\alpha_1)\ldots(z-\alpha_m), \quad \text{where} \quad a_0 \neq 0,$$

of $f(z)$ implies that
$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{m} \frac{1}{z-\alpha_j},$$

hence that
$$\int_0^1 \log \left| \sum_{j=1}^{m} \frac{1}{e^{2\pi i\vartheta} - \alpha_j} \right| d\vartheta = \log \frac{M(f')}{M(f)}.$$

Therefore theorems 1 and 3, and so also theorems 2 and 3, are likewise equivalent.

It suffices then to prove theorem 1. This proof is indirect. It is based on a continuity property of the measure $M(f)$ which has some interest in itself.

2. This continuity property is as follows.

LEMMA 1. *Let* $f(z)$ *be a fixed polynomial, and let* $\{f_k(z)\}$ *be an infinite sequence of polynomials of bounded degrees such that*

$$\lim_{k\to\infty} L(f_k - f) = 0. \tag{4}$$

Then
$$\lim_{k\to\infty} M(f_k) = M(f).$$

Remark. In the hypothesis (4), the length $L(f_k - f)$ may also be replaced by the measure $M(f_k - f)$, as is obvious from (3). It would be interesting to decide whether the lemma remains valid when the restriction on the degrees of the polynomials $f_k(z)$ is omitted.

Proof of lemma 1. The assertion follows immediately from (3) if the limit polynomial $f(z)$ vanishes identically; let this trivial case be excluded. Without loss of generality, finitely many of the polynomials $f_k(z)$ may be omitted. Hence it may be assumed that the polynomials $f(z)$ and $f_k(z)$ have the explicit form

$$f(z) = a_0 z^m + a_1 z^{m-1} + \ldots + a_m = a_0(z-\alpha_1)\ldots(z-\alpha_m), \quad \text{where} \quad a_0 \neq 0,$$

and

$$f_k(z) = a_{k0} z^{m_k} + a_{k1} z^{m_k - 1} + \ldots + a_{km_k} = a_{k0}(z-\alpha_{k1})\ldots(z-\alpha_{km_k}), \quad \text{where} \quad a_{k0} \neq 0,$$

and where
$$m \leqslant m_k \leqslant n \quad \text{for all } k.$$

Here n is a certain constant.

The hypothesis (4) implies that

$$\lim_{k\to\infty} f_k(z) = f(z)$$

uniformly in every bounded set in the complex z-plane. Hence, by a theorem due to Hurwitz (see, for example, Marden 1949, p. 4) it is possible to number the zeros of the polynomials $f_k(z)$ such that

$$\lim_{k\to\infty} \alpha_{kj} = \alpha_j \quad (j = 1, 2, \ldots, m), \tag{5}$$

but that the remaining zeros

$$\alpha_{km+1}, \alpha_{km+2}, \ldots, \alpha_{km_k}$$

tend to infinity with k for all those suffixes k for which the difference

$$l_k = m_k - m$$

is positive.

Denote by K, and K', arbitrary infinite sequences of distinct suffixes k for which $l_k = 0$, or $l_k > 0$, respectively.

First, let k tend to infinity over any sequence K. Then, by (4),

$$\lim a_{k0} = a_0,$$

whence, by (5),

$$M(f_k) = |a_{k0}| \prod_{j=1}^{m} \max(1, |\alpha_{kj}|) \to |a_0| \prod_{j=1}^{m} \max(1, |\alpha_j|) = M(f).$$

Secondly, assume that k tends to infinity over any sequence K'. For such suffixes the expression

$$(-1)^{l_k} a_{kl_k}/a_{k0}$$

is that elementary symmetric function of the zeros

$$\alpha_{k1}, \alpha_{k2}, \ldots, \alpha_{km_k}$$

of $f_k(z)$ in which every term

$$\alpha_{ki_1}\alpha_{ki_2}\ldots\alpha_{ki_{l_k}} \quad (1 \leqslant i_1 < i_2 < \ldots < i_{l_k} \leqslant m_k)$$

is a product of l_k factors. All the l_k factors of exactly one term

$$\alpha_{km+1}\alpha_{km+2}\ldots\alpha_{km_k}$$

of this symmetric function tend to infinity with k, while in the remaining terms at least one factor remains bounded. It follows that

$$|\alpha_{km+1}\alpha_{km+2}\ldots\alpha_{km_k}| \sim |a_{kl_k}/a_{k0}|.$$

Now a_{kl_k} is the coefficient of z^m in $f_k(z)$, and so, by (4),

$$a_{kl_k} \to a_0,$$

hence $$\lim |a_{kl_k}| \prod_{j=1}^{m} \max(1, |\alpha_{kj}|) = |a_0| \prod_{j=1}^{m} \max(1, |\alpha_j|) = M(f).$$

On the other hand, $$\prod_{j=m+1}^{m_k} \max(1, |\alpha_{kj}|) = \prod_{j=m+1}^{m_k} |\alpha_{kj}|$$

for all sufficiently large k because the factors tend to infinity. It follows thus, finally, that as $k \in K'$ tends to infinity,

$$M(f_k) = |a_{k0}| \prod_{j=1}^{m_k} \max(1, |\alpha_{kj}|) = \left|\frac{a_{k0}}{a_{kl_k}}\right| \prod_{j=m+1}^{m_k} |\alpha_{kj}| \times |\alpha_{kl_k}| \prod_{j=1}^{m} \max(1, |\alpha_{kj}|) \to M(f),$$

because the first factor on the right-hand side tends to 1 and the second factor to $M(f)$. The assertion is therefore true in either case and hence generally.

3. *The proof of theorem* 1 *is indirect.* We shall assume that there exists at least one polynomial $g(z)$, say of the exact degree m, such that

$$M(g') > mM(g). \tag{6}$$

From this hypothesis, after a long and involved reasoning, we shall derive a contradiction.

Let m be the smallest degree for which there exists a polynomial $g(z)$ with the property (6); there is no difficulty in showing that

$$m \geqslant 2.$$

Denote by S the set of all polynomials $f(z) \not\equiv 0$ at most of degree m, and for such polynomials put

$$Q(f) = M(f')/M(f).$$

Further let

$$\Lambda = \sup_{f \in S} Q(f). \tag{7}$$

It is easily proved that

$$m < \Lambda \leqslant 2^m m. \tag{8}$$

The left-hand inequality follows from the existence of $g(z)$. Next, it is obvious that

$$L(f') \leqslant mL(f) \quad \text{if} \quad f(z) \in S.$$

Therefore, by (3),

$$M(f') \leqslant L(f') \leqslant mL(f) \leqslant 2^m mM(f), \quad \text{hence} \quad Q(f) \leqslant 2^m m \quad \text{if} \quad f(z) \in S,$$

giving the right-hand inequality (8).

If c is any constant,

$$L(cf) = |c|\, L(f) \quad \text{and} \quad Q(cf) = Q(f).$$

Therefore, in the definition (7) of Λ, the least upper bound need only be extended over those polynomials $f(z)$ in S which are normed by

$$L(f) = 1.$$

Since Λ is bounded, it follows then finally that there exists an infinite sequence of polynomials

$$\Sigma = \{f_k(z)\}$$

in S with the properties

$$\lim_{k \to \infty} Q(f_k) = \Lambda \quad \text{and} \quad L(f_k) = 1 \quad \text{for all } k. \tag{9}$$

4. Each polynomial $f_k(z)$ in Σ may be written explicitly as

$$f_k(z) = a_{k0}z^m + a_{k1}z^{m-1} + \dots + a_{km},$$

where

$$L(f_k) = |a_{k0}| + |a_{k1}| + \dots + |a_{km}| = 1.$$

We represent this polynomial by the point $(a_{k0}, a_{k1}, \ldots, a_{km})$ in complex $(m+1)$-dimensional space; then all these points are bounded. By Weierstrass's theorem it is thus possible to select an infinite subsequence of Σ,

$$\Sigma' = \{f_{k_l}(z)\} \text{ say,} \quad \text{where} \quad k_1 < k_2 < k_3 < \ldots,$$

such that the polynomials $f_{k_l}(z)$ in Σ' converge to a certain limit polynomial

$$f_0(z) = a_{00}z^m + a_{01}z^{m-1} + \ldots + a_{0m}$$

in the sense that

$$\lim_{l\to\infty} L(f_{k_l} - f_0) = 0. \tag{10}$$

This formula is equivalent to the $m+1$ limit relations

$$\lim_{l\to\infty} a_{k_l j} = a_{0j} \quad (j = 0, 1, \ldots, m)$$

for the coefficients. In particular,

$$L(f_0) = \lim_{l\to\infty} L(f_{k_l}) = 1,$$

which means that $f_0(z)$ is not identically zero. It is also evident that

$$\lim_{l\to\infty} L(f'_{k_l} - f'_0) = 0 \tag{11}$$

because

$$L(f'_{k_l} - f'_0) \leqslant mL(f_{k_l} - f_0) \quad (l = 1, 2, 3, \ldots).$$

5. Finitely many terms of the sequence Σ' may always be omitted. There is then no loss of generality, by (8) and (9), in assuming that

$$m \leqslant Q(f_{k_l}) \leqslant \Lambda \quad (l = 1, 2, 3, \ldots).$$

Also, by (3) and the second formula (9),

$$2^{-m} \leqslant M(f_{k_l}) \leqslant 1 \quad (l = 1, 2, 3, \ldots),$$

whence further

$$2^{-m}m \leqslant M(f'_{k_l}) = M(f_{k_l})\,Q(f_{k_l}) \leqslant \Lambda \quad (l = 1, 2, 3, \ldots).$$

These inequalities show that both sequences of real numbers

$$\{M(f_{k_l})\} \quad \text{and} \quad \{M(f'_{k_l})\}$$

have finite positive lower and upper bounds. Hence Weierstrass's theorem allows to select in Σ' an infinite subsequence

$$\Sigma'' = \{f_{k^{(l)}}(z)\}, \quad \text{where} \quad k' < k'' < k''' < \ldots,$$

such that both limits

$$\lim_{l\to\infty} M(f_{k^{(l)}}) \quad \text{and} \quad \lim_{l\to\infty} M(f'_{k^{(l)}})$$

exist and are positive and finite. The polynomials in Σ'' still satisfy the relations (10) and (11). Hence lemma 1 implies that

$$M(f_0) = \lim_{l\to\infty} M(f_{k^{(l)}}) \quad \text{and} \quad M(f'_0) = \lim_{l\to\infty} M(f'_{k^{(l)}}),$$

and that therefore

$$Q(f_0) = M(f'_0)/M(f_0) = \lim_{l\to\infty} M(f'_{k^{(l)}})/M(f_{k^{(l)}}) = \lim_{l\to\infty} Q(f_{k^{(l)}}) = \lim_{k\to\infty} Q(f_k) = \Lambda > m.$$

The polynomial $g(z) \equiv f_0(z)$ possesses thus the property (6) and so, by the definition of m, has the exact degree m. Its highest coefficient a_{00} is therefore distinct from zero.

Put
$$F(z) = a_{00}^{-1} f_0(z),$$
so that $F(z)$ has the exact degree m and the highest coefficient 1. By the homogeneity of $Q(f)$,
$$Q(F) = Q(f_0) = \Lambda.$$
It follows therefore the basic inequality,
$$Q(f) \leqslant Q(F) \quad \textit{for all polynomials } f(z) \textit{ in } S. \tag{12}$$

6. The polynomial $F(z)$ and its derivative may be factorized in the form
$$F(z) = (z-A_1)\dots(z-A_m),$$
$$F'(z) = m(z-B_1)\dots(z-B_{m-1});$$
here the zeros A_j and B_j need not all be distinct, and some of them may vanish. The notation may be chosen such that, say,
$$|A_j| > 1 \quad \text{if} \quad 1 \leqslant j \leqslant r, \quad A_j = 0 \quad \text{if} \quad r+1 \leqslant j \leqslant r+s,$$
$$0 < |A_j| \leqslant 1 \quad \text{if} \quad r+s+1 \leqslant j \leqslant m; \tag{13}$$
$$|B_j| > 1 \quad \text{if} \quad 1 \leqslant j \leqslant R, \quad |B_j| \leqslant 1 \quad \text{if} \quad R+1 \leqslant j \leqslant m-1. \tag{14}$$
Then
$$M(F) = |A_1A_2\dots A_r| \quad \text{and} \quad M(F') = m\,|B_1B_2\dots B_R|,$$
so that
$$Q(F) = m\left|\frac{B_1B_2\dots B_R}{A_1A_2\dots A_r}\right| = \Lambda > m. \tag{15}$$

From this formula it follows in particular that
$$R \geqslant 1$$
because otherwise $Q(F) \leqslant m$. But this implies that also
$$r \geqslant 1.$$
For when $r = 0$, all zeros of $F(z)$ lie inside or on the unit circle. By the theorem of Lucas (see, for example, Marden 1949, p. 14), the same is then true for the zeros of $F'(z)$. Hence $R = 0$, against what has just been proved.

7. Denote now by
$$Z_1, Z_2, \dots, Z_{r+s}$$
$r+s$ independent complex variables, and put
$$f(z) = g(z)\,h(z), \quad \text{where} \quad g(z) = \prod_{j=1}^{r+s}(z-Z_j) \quad \text{and} \quad h(z) = \prod_{j=r+s+1}^{m}(z-A_j). \tag{16}$$
This polynomial $f(z)$ has again the degree m and the highest coefficient 1, and it coincides with $F(z)$ if
$$Z_j = A_j \quad \text{when} \quad 1 \leqslant j \leqslant r+s.$$
Its derivative allows the factorization
$$f'(z) = m\prod_{j=1}^{m-1}(z-W_j), \tag{17}$$

where $W_1, W_2, \ldots, W_{m-1}$ are the branches of the algebraic function

$$W(Z) = W(Z_1, Z_2, \ldots, Z_{r+s})$$

of $(Z) = (Z_1, Z_2, \ldots, Z_{r+s})$ defined by

$$f'(W) = 0.$$

By Hurwitz's theorem, there exists a neighbourhood

$$U: |Z_1 - A_1| < \delta, \quad |Z_2 - A_2| < \delta, \quad \ldots, \quad |Z_{r+s} - A_{r+s}| < \delta$$

of the point $(A) = (A_1, A_2, \ldots, A_{r+s})$ such that the following relations hold when the point (Z) lies in this neighbourhood,

(A): $\quad W_1, W_2, \ldots, W_{m-1}$ are continuous functions of (Z);

(B): $\quad |Z_j| > 1 \quad \text{if} \quad 1 \leqslant j \leqslant r \quad \text{and} \quad |Z_j| < 1 \quad \text{if} \quad r+1 \leqslant j \leqslant r+s;$

(C): $\quad |W_j| > 1 \quad \text{if} \quad 1 \leqslant j \leqslant R;$

(D): $W_1 = B_1,\ W_2 = B_2,\ \ldots,\ W_{m-1} = B_{m-1} \quad \text{if} \quad Z_1 = A_1,\ Z_2 = A_2,\ \ldots,\ Z_{r+s} = A_{r+s}.$

It is quite possible that for some points (Z) in U also one or more of the numbers $|W_{R+1}|, |W_{R+2}|, \ldots, |W_{m-1}|$ are greater than 1.

Independent of whether such points (Z) exist or not, it is obvious that

$$M(f) = |Z_1 Z_2 \ldots Z_r| \quad \text{and} \quad M(f') \geqslant m\,|W_1 W_2 \ldots W_R|,$$

hence that

$$Q(f) \geqslant m \left| \frac{W_1 W_2 \ldots W_R}{Z_1 Z_2 \ldots Z_r} \right|.$$

On combining this inequality with the formulae (12) and (15), we find that

$$\left| \frac{W_1 W_2 \ldots W_R}{Z_1 Z_2 \ldots Z_r} \right| \leqslant \left| \frac{B_1 B_2 \ldots B_R}{A_1 A_2 \ldots A_r} \right| \quad \text{if} \quad (Z) \in U. \tag{18}$$

8. Put now

$$X(Z) = X(Z_1, Z_2, \ldots, Z_{r+s}) = \frac{W_1 W_2 \ldots W_R}{Z_1 Z_2 \ldots Z_r}.$$

By the assumptions $(A) - (D)$, $X(Z)$ is a continuous branch of an algebraic function of (Z) when (Z) lies in U. Also

$$X(A) = X(A_1, A_2, \ldots, A_{r+s}) = \frac{B_1 B_2 \ldots B_R}{A_1 A_2 \ldots A_r},$$

so that, by (18),

$$|X(Z)| \leqslant |X(A)| \quad \text{if} \quad (Z) \in U. \tag{19}$$

Assume, for the moment, that $X(Z)$ is *not* a constant. There are then $r+s$ constants $\beta_1, \beta_2, \ldots, \beta_{r+s}$ satisfying

$$|\beta_j| < \delta \quad \text{for} \quad 1 \leqslant j \leqslant r+s$$

such that $\quad X(A_1 + \beta_1, A_2 + \beta_2, \ldots, A_{r+s} + \beta_{r+s}) \neq X(A_1, A_2, \ldots, A_{r+s}).$

The function $\quad x(t) = X(A_1 + \beta_1 t, A_2 + \beta_2 t, \ldots, A_{r+s} + \beta_{r+s} t)$

of the single complex variable t is then likewise not a constant. The point

$$(A + \beta t) = (A_1 + \beta_1 t, A_2 + \beta_2 t, \ldots, A_{r+s} + \beta_{r+s} t)$$

lies in U when t belongs to the circle

$$|t| < 1.$$

For such t, $x(t)$ is a non-constant continuous branch of an algebraic function of t which at the point $t = 0$ assumes the value

$$x(0) = X(A).$$

The point $t = 0$ may be a regular point or a branch point of $x(t)$; in either case there is a positive integer N such that $x(u^N)$ is regular and non-constant in a certain small circle

$$V: |u| \leqslant \epsilon,$$

where $0 < \epsilon < 1$. By the maximum principle for regular analytic functions, this implies that

$$|x(0)| < \max_{u \in V} |x(u^N)|.$$

Thus there exists a complex number t_0 satisfying

$$|t_0| < 1, \quad |x(t_0)| > |x(0)| = |X(A)|.$$

But then the point $(A + t_0 \beta)$ lies in U, and

$$|X(A + t_0 \beta)| > |X(A)|,$$

contrary to (19).

This contradiction proves that our assumption was false and that

$X(Z)$ is a constant.

This property holds, in the first instance, when (Z) lies in U; but if $X(Z)$ is defined on its Riemann manifold by analytic continuation, then it remains always true. Hence a constant C exists such that

$$W_1 W_2 \dots W_R = C Z_1 Z_2 \dots Z_r \quad \textit{identically in} \quad Z_1, Z_2, \dots, Z_{r+s}. \tag{20}$$

9. We now replace the $r+s$ variables $Z_1, Z_2, \dots, Z_{r+s}$ by a single independent variable Z by defining $Z_1, Z_2, \dots, Z_r$ as the roots of the equation

$$k(z) = 0, \quad \text{where} \quad k(z) = z^r + z^{r-1} + z^{r-2} + \dots + z + Z, \tag{21}$$

and by putting

$$Z_{r+1} = Z_{r+2} = \dots = Z_{r+s} = 1.$$

Then $f(z)$ becomes the polynomial

$$f(z) = k(z)\, l(z),$$

where

$$l(z) = (z-1)^s h(z) = (z-1)^s \prod_{j=r+s+1}^{m} (z - A_j).$$

Here it is obvious, from (13), that

$$l(0) \neq 0. \tag{22}$$

From its definition, $f(z)$ has the exact degree m and the highest coefficient 1; all the other coefficients are linear polynomials in Z. It follows that its derivative $f'(z)$ has the highest coefficient m while all its other coefficients are again linear polynomials in Z.

The lowest coefficient of $f'(z)$ has the value $f'(0)$ and can be obtained as follows. On putting $z = 0$ in

$$f'(z) = k'(z)\,l(z) + k(z)\,l'(z)$$

and using

$$k(0) = Z, \quad k'(0) = 1,$$

we find that

$$f'(0) = l(0) + l'(0)\,Z. \tag{23}$$

10. The proof of the theorem can now be concluded as follows.

Since $Z_1, Z_2, \ldots, Z_r$ are the zeros of $k(z)$, evidently

$$Z_1 Z_2 \ldots Z_r = (-1)^r Z. \tag{24}$$

Similarly, $W_1, W_2, \ldots, W_{m-1}$ are the zeros of $f'(z)$ which has the highest coefficient m and the lowest coefficient (23); therefore also

$$W_1 W_2 \ldots W_{m-1} = \frac{(-1)^{m-1}}{m}\,(l(0) + l'(0)\,Z). \tag{25}$$

From the form of the equation $f'(z) = 0$, its roots $W_1, W_2, \ldots, W_{m-1}$ are continuous functions of Z. Let now Z tend to zero. It follows then from (20) and (24) that

$$\lim_{Z \to 0} W_1 W_2 \ldots W_R = 0$$

and hence also that

$$\lim_{Z \to 0} W_1 W_2 \ldots W_{m-1} = 0.$$

On the other hand, from (25),

$$\lim_{Z \to 0} W_1 W_2 \ldots W_{m-1} = \frac{(-1)^{m-1}}{m}\,l(0),$$

so that by (22) the limit is distinct from zero.

This contradiction shows that the hypothesis at the beginning of the proof in § 3 is false and so concludes the proof of theorem 1.

11. A slight change of the proof just given enables us also to decide for which polynomials $f(z)$ of exact degree m we may have

$$M(f') = mM(f).$$

This equation certainly holds when the absolute values of all the zeros of $f(z)$ and hence, by Lucas's theorem, also those of all the zeros of $f'(z)$, are at most equal to 1. For in this special case $M(f)$ and $M(f')$ become simply the absolute values of the highest coefficients of these two polynomials.

This exceptional case is, however, the only one with equality.

For assume there exists a polynomial $F(z)$ of exact degree m which has at least one zero of absolute value greater than 1 and is such that

$$M(F') = mM(F).$$

Then, in the notation of § 6,

$$Q(F) = m\left|\frac{B_1 B_2 \ldots B_R}{A_1 A_2 \ldots A_r}\right| = m,$$

and hence not only $r \geqslant 1$, but also $R \geqslant 1$. Further, by theorem 1, it is again true that

$$Q(f) \leqslant Q(F) \quad \text{for all polynomials } f(z) \text{ in } S.$$

But this means that the whole proof in §§ 6–10 can be repeated without any essential changes and again leads to a contradiction.

By the equivalence of the three theorems it therefore also follows that equality holds in theorem 2 if and only if none of the numbers $|\alpha_j|$ and $|\beta_j|$ exceeds ρ; and that it holds in theorem 3 if and only if all the numbers $|\alpha_j|$ are at most 1. Thus, in particular,

$$\int_0^1 \log\left|\sum_{j=1}^{m} \frac{1}{e^{2\pi i\vartheta}-\alpha_j}\right| d\vartheta \begin{cases} = \log m & \text{if} \quad |\alpha_j| \leqslant 1 \quad \text{for} \quad 1 \leqslant j \leqslant m, \\ < \log m & \text{otherwise.} \end{cases}$$

12. The method of this paper is rather general and possibly may have applications to other extremal problems. It is also probable that theorem 1 may have applications, e.g. in the theory of entire functions.

There are other problems on $M(f)$ that seem worthy of study. Two of these are of particular interest because they would have applications in the theory of Diophantine approximations.

(I) For given m, to find the best possible constant c_m such that, if $f(z)$ and $g(z)$ are any two polynomials at most of degree m, then

$$M(f+g) \leqslant c_m(M(f)+M(g)).$$

(II) For given m and n, to find the best possible constant C_{mn} such that, if $f_1(z), \ldots, f_n(z)$ are any n polynomials at most of degree m, then

$$\sum_{h=1}^{n}\sum_{k=1}^{n} M(f_h-f_k) \leqslant C_{mn}\sum_{h=1}^{n} M(f_h).$$

From the inequality (3) it is easy to deduce that

$$c_m \leqslant 2^m, \quad C_{mn} \leqslant 2^m(n-1),$$

but these estimates are nearly certainly too large. It seems probable that in both cases the extrema are again attained when all the polynomials have only zeros of absolute values not exceeding 1.

I finally thank my colleague A. Stone for reading the manuscript and suggesting many improvements.

Postscript (29 *April* 1961)

The referee for the Royal Society has made the following important remark about my paper:

'The integral in theorem 3 is continuous for all values of each α and subharmonic in each individual α provided α does not lie on the boundary of the unit circle. Hence it suffices to prove the theorem if each α does lie on the boundary of the unit circle. But then the formula is obvious by the equivalence of theorems 2 and 3.'

There is thus a simpler approach to the results of my paper. However, the method I am using has interest in itself, and I have now succeeded in applying it to several other problems on polynomials.

References

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