A correction to the paper

An interpolation Series for continuous functions of a p-adic variable.

By *K. Mahler* in Manchester.

Dr. M. A. Maurice of Amsterdam has drawn my attention to a serious error in the proof of Theorem 2 of my paper in vol. **199** (1958), p. 23—34, of this journal. It is stated on p. 29 that one may assume, without loss of generality, that there is a positive integer s such that

$$\left|\frac{a_n}{n}\right|_p \leqq \frac{1}{p} \quad \text{if} \quad p^s \nmid n \text{ and } n \geqq p^s.$$

This statement, unfortunately, is false and invalidates the proof.

I have not succeeded in altering this proof so as to make it correct. Therefore I shall give here a new proof based on entirely different ideas.

The assertion to be proved is as follows.

Theorem 2. *Let $\{a_n\}$ be a p-adic null sequence. If the limit*

$$\lambda = \lim_{|x|_p \to 0} \sum_{n=1}^{x} \frac{a_n}{n} \binom{x-1}{n-1}$$

extended over all elements $x \neq 0$ of J exists, then:

(i) $$\lim_{n \to \infty} \frac{a_n}{n} = 0,$$

and

(ii) $$\lambda = \sum_{n=1}^{\infty} \frac{a_n}{n} \binom{-1}{n-1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a_n}{n}.$$

The difficulty lies in the proof of (i). Once this relation has been obtained, the second assertion (ii) follows immediately from the Lemma 4, p. 31, of my paper.

The new proof of (i) runs as follows:

Put

$$f(x) = \sum_{n=1}^{\infty} a_n \binom{x}{n}.$$

Since $\{a_n\}$ is a null sequence, $f(x)$ is a continuous function of $x \in I$; moreover,

$$f(0) = 0.$$

Therefore, when $x = 0$ lies in J,

$$\frac{f(x) - f(0)}{x} = \frac{f(x)}{x} = \sum_{n=1}^{x} \frac{a_n}{x}\binom{x}{n} = \sum_{n=1}^{x} \frac{a_n}{n}\binom{x-1}{n-1}.$$

The hypothesis of Theorem 2 is therefore equivalent to

$$(1) \qquad \lim_{\substack{|x|_p \to 0 \\ x \neq 0 \\ x \in J}} \frac{f(x) - f(0)}{x} = \lim_{\substack{|x|_p \to 0 \\ x \neq 0 \\ x \in J}} \frac{f(x)}{x} = \lambda.$$

We first prove that this limit formula implies the stronger equation

$$(2) \qquad \lim_{\substack{|\xi|_p \to 0 \\ \xi \neq 0 \\ \xi \in I}} \frac{f(\xi) - f(0)}{\xi} = \lim_{\substack{|\xi|_p \to 0 \\ \xi \neq 0 \\ \xi \in I}} \frac{f(\xi)}{\xi} = \lambda,$$

and so means that $f(x)$ is differentiable at $x = 0$ and has the derivative

$$f'(0) = \lambda.$$

The formula (1) holds if and only if, given any positive integer s, there exists a second positive integer t such that

$$(3) \qquad \left|\frac{f(x)}{x}\right|_p \leqq p^{-s} \quad \text{if} \quad x \in J, \quad 0 < |x|_p \leqq p^{-t}.$$

We prove (2) by showing that then also

$$(4) \qquad \left|\frac{f(\xi)}{\xi}\right|_p \leqq p^{-s} \quad \text{if} \quad \xi \in I, \quad 0 < |\xi|_p \leqq p^{-t}.$$

For let ξ be an arbitrary element of I satisfying

$$0 < |\xi|_p \leqq p^{-t}.$$

If $f(\xi) = 0$, (4) certainly holds; hence we assume that

$$f(\xi) \neq 0.$$

By the hypothesis, $f(x)$ is continuous at $x = \xi$. Therefore a positive integer u exists such that

$$|f(\eta) - f(\xi)|_p < |f(\xi)|_p \quad \text{if} \quad \eta \in I, \quad |\eta - \xi|_p \leqq p^{-u}$$

and hence also

$$|f(\eta)|_p = |f(\xi)|_p \quad \text{if} \quad \eta \in I, \quad |\eta - \xi|_p \leqq p^{-u}.$$

Since ξ is a p-adic integer, we can now determine a positive integer x such that

$$|x - \xi|_p \leqq \min\left(p^{-u}, \ \frac{1}{p}|\xi|_p\right).$$

Then x satisfies both the equations

$$|f(x)|_p = |f(\xi)|_p \quad \text{and} \quad |x|_p = |\xi|_p,$$

and furthermore

$$0 < |x|_p \leqq p^{-t}.$$

It follows then that

$$\left|\frac{f(\xi)}{\xi}\right|_p = \left|\frac{f(x)}{x}\right|_p \leqq p^{-s},$$

and so the assertion (4) is a direct consequence of (3). —

From now on we may assume that

$$f(0) = 0 \text{ and } f'(0) = \lambda,$$

because the hypothesis of Theorem 2 implies these two formulae. Define a function $g(x)$ by

$$g(x) = \begin{cases} 0 & \text{if } x = 0, \\ \dfrac{f(x) - \lambda x}{x} & \text{if } x \neq 0, \end{cases} \quad x \in I.$$

As the quotient of two continuous functions $g(x)$ is certainly continuous for all $x \neq 0$; but it is also continuous when $x = 0$ because

$$\lim_{|x|_p \to 0} g(x) = \lim_{|x|_p \to 0} \left(\frac{f(x)}{x} - \lambda \right) = \lambda - \lambda = 0 = g(0).$$

Hence $g(x)$ is continuous everywhere on I. But then, by Theorem 1 of my paper, it can be written as an interpolation series

$$g(x) = \sum_{n=0}^{\infty} b_n \binom{x}{n}$$

where $\{b_n\}$ is again a p-adic null sequence.

Now, identically in x,

$$f(x) = \lambda x + x g(x) = \lambda \binom{x}{1} + \sum_{n=0}^{\infty} b_n x \binom{x}{n},$$

and here

$$x \binom{x}{n} = (x - n) \binom{x}{n} + n \binom{x}{n} = (n+1) \binom{x}{n+1} + n \binom{x}{n},$$

so that

$$f(x) = \lambda \binom{x}{1} + \sum_{n=0}^{\infty} b_n \left\{ (n+1) \binom{x}{n+1} + n \binom{x}{n} \right\}.$$

On the other hand, also

$$f(x) = \sum_{n=1}^{\infty} a_n \binom{x}{n},$$

and the interpolation series for $f(x)$ is unique. Hence, on equating the coefficients of identical binomial coefficients, it follows that

$$a_1 = \lambda + b_0 + b_1; \quad a_2 = 2(b_1 + b_2); \quad a_3 = 3(b_2 + b_3); \ldots.$$

Generally, for $n \geqq 2$,

$$a_n = n(b_{n-1} + b_n).$$

But then

$$\lim_{n \to \infty} \left| \frac{a_n}{n} \right|_p = \lim_{n \to \infty} | b_{n-1} + b_n |_p = 0,$$

whence the assertion (i).

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