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On the approximation of real numbers by roots of integers

by

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To J. G. van der Corput on his 75th birthday

Let A be the set of all numbers $a > 1$ for which none of the powers $a, a^2, a^3, \ldots$ is an integer. For every positive integer n there exists at least one integer g_n which is closest to a^n and thus satisfies the inequality

$$|a^n - g_n| \leqslant 1/2.$$

We are here concerned with the lower limit

$$P(a) = \liminf_{n\to\infty} |a^n - g_n|^{1/n}$$

which trivially has the property

$$0 \leqslant P(a) \leqslant 1 \quad \text{for all } a \in A.$$

A few years ago, one of us (Mahler, 1957) proved that

$$P(a) = 1 \text{ if } a \text{ is any rational number in } A.$$

One can further show that there are irrational algebraic numbers $a \in A$ for which $P(a) = 1$; e.g. the number $\frac{1}{2}(2+\sqrt{3}+\sqrt{3+4\sqrt{3}})$ is of this kind. It is also well known that there exist algebraic numbers a in A for which

$$0 < P(a) < 1;$$

e.g. the number $\frac{1}{2}(1+\sqrt{5})$ has this property.

In the present note, the following three results will be proved.

(a) *If $P(a) = 0$, then a is transcendental.*

(b) *In every neighbourhood of every number $x > 1$ there exist non-countably many $a \in A$ for which $P(a) = 0$.*

(c) *For almost all a in A, $P(a) = 1$; thus there are transcendental numbers with this property.*

Proof of (a). Let a be an algebraic number of degree m, and let

$$f(x) = a_0 x^m + a_1 x^{m-1} + \dots + a_m = a_0 \prod_{\mu=0}^{m-1} (x - a^{(\mu)})$$

be a primitive irreducible polynomial with integral coefficients of which $a = a^{(0)}$ is a zero. For each n the product

$$a_0^n \prod_{\mu=0}^{m-1} (a^{(\mu)n} - g_n) = p_n$$

say, is an integer. This integer is distinct from zero because $a^n \neq g_n$ and hence also

$$a^{(\mu)n} - g_n \neq 0 \qquad (\mu = 0, 1, \dots, m-1).$$

Therefore

$$|p_n| \geqslant 1.$$

Next $a > 1$, hence $g_n \geqslant 1$, and therefore

$$|a^{(\mu)n} - g_n| \leqslant g_n (|a^{(\mu)}| + 1)^n,$$

so that

$$|a_0|^n \prod_{\mu=1}^{m-1} |a^{(\mu)n} - g_n| \leqslant g_n^{m-1} \Big\{ |a_0| \prod_{\mu=1}^{m-1} (|a^{(\mu)}| + 1) \Big\}^n.$$

Here

$$g_n \leqslant a^n + \tfrac{1}{2} < (2a)^n$$

and so finally

$$1 \leqslant |p_n| \leqslant |a^n - g_n| \Big\{ (2a)^{m-1} |a_0| \prod_{\mu=1}^{m-1} (|a^{(\mu)}| + 1) \Big\}^n.$$

There exists then a constant $c > 1$ depending only on a such that

$$|a^n - g_n| \geqslant c^{-n} \qquad \text{for all } n,$$

proving that $P(a) > 0$. Conversely, if $P(a) = 0$, then a necessarily is transcendental.

Proof of (b). Let $x > 1$, $0 < \varepsilon < \frac{1}{2}(x-1)$. We show that there is a sequence of positive integers

$$1 = n_0 < n_1 < n_2 < \dots < n_k < \dots$$

depending on x, but not on ε, with the following property:

Given any sequence $\{\eta_k\}$ with η_k equal to either 0 or 1 (briefly, an η-sequence), there is a real number a, where

(1) $$0 < x - a < \varepsilon,$$

such that

(2) $$\lim_{k\to\infty} \{ |a^{n_k} - g_{n_k}|^{1/n_k} - \eta_k \} = 0.$$

Clearly if $\eta_k = 0$ for infinitely many values of k, then $P(\alpha) = 0$; and if $\{\eta_k\}$, $\{\eta_k'\}$ are two essentially different η-sequences, i.e. such that $\eta_k \neq \eta_k'$ for infinitely many k, then the corresponding real numbers α and α' are distinct. Since there are non-countably many essentially different η-sequences, we obtain non-countably many α with $P(\alpha) = 0$ in the (left) ε-neighbourhood of x, hence also non-countably many $\alpha \in A$ with this property.

For the proof take any increasing sequence n_k which satisfies the condition

$$(3)\qquad -\frac{1}{n_k}\log\left(1-2\left(\frac{1+x}{2}\right)^{-n_k}\right) \leqslant \frac{x-1}{x+1}2^{-kn_{k-1}}x^{-n_{k-1}}/n_{k-1} \quad \text{for} \quad k>0.$$

The condition is clearly satisfied if n_k increases sufficiently rapidly.

Let $\{\eta_k\}$ be an arbitrary η-sequence and $\varepsilon > 0$; we may assume $\varepsilon < \frac{1}{2}$. Determine $K > 1$ so that

$$(4)\qquad \prod_{k=K}^{\infty}(1-2^{-k}) > 1-\frac{\varepsilon}{x}.$$

We define now $x = x_0 \geqslant x_1 \geqslant x_2 \geqslant \ldots$ as follows: For $0 \leqslant k < K$ we set $x_k = x$. Suppose that for some $k \geqslant K$, x_{k-1} has already been determined so that

$$(5)\qquad \tfrac{1}{2}(1+x)+2^{-k}(x-1) \leqslant x_{k-1} \leqslant x.$$

Set

$$(6)\qquad x_{k-1}^{n_k} = a_k+\lambda_k, \qquad a_k \text{ integer}, \ 1 \leqslant \lambda_k < 2.$$

We then define

$$(7)\qquad x_k = (a_k+\tfrac{1}{2}\eta_k+2^{-kn_k})^{1/n_k}.$$

Clearly $\frac{1}{2}\eta_k+2^{-kn_k} \leqslant 1 \leqslant \lambda_k$, $x_k \leqslant x_{k-1}$, and so the second half of inequality (5) is satisfied. We now show that also

$$x_k \geqslant \tfrac{1}{2}(1+x)+2^{-k-1}(x-1).$$

Set

$$(8)\qquad x_k = x_{k-1}\left(1-\frac{\delta_k}{n_k}\right).$$

By (6) and (7) we have for $k \geqslant K$

$$a_k+\tfrac{1}{2}\eta_k+2^{-kn_k} = (a_k+\lambda_k)\left(1-\frac{\delta_k}{n_k}\right)^{n_k},$$

or setting

$$\zeta_k = \lambda_k - \tfrac{1}{2}\eta_k - 2^{-kn_k}, \qquad 0 \leqslant \zeta_k < 2,$$

$$1 - \zeta_k x_{k-1}^{-n_k} = \left(1 - \frac{\delta_k}{n_k}\right)^{n_k} < e^{-\delta_k}.$$

By (5) therefore

$$(9) \qquad 0 \leqslant \delta_k \leqslant -\log\left(1 - \zeta_k\left(\frac{1+x}{2}\right)^{-n_k}\right) < -\log\left(1-2\left(\frac{1+x}{2}\right)^{-n_k}\right),$$

hence by (5) and (8),

$$x_k \geqslant \tfrac{1}{2}(x+1) + 2^{-k}(x-1)\left(1 + \frac{1}{n_k}\log\left(1-2\left(\frac{1+x}{2}\right)^{-n_k}\right)\right)$$
$$\geqslant \tfrac{1}{2}(x+1) + 2^{-k-1}(x-1),$$

since

$$(10) \qquad \frac{1}{n_k}\log\left(1-2\left(\frac{1+x}{2}\right)^{-n_k}\right) \geqslant -\frac{x-1}{x+1}\,2^{-kn_{k-1}}x^{-n_{k-1}}/n_{k-1}$$
$$> -\frac{x-1}{x+1}\,2^{-k-1} \geqslant -\left(2 + \frac{x+1}{x-1}\,2^k\right)^{-1}$$

by (3). Thus (5) is proved for x_k.

From (5) and the monotonity of x_k it follows that $\alpha = \lim_{k\to\infty} x_k$ exists and $\frac{1}{2}(1+x) \leqslant \alpha \leqslant x$. From (8) and (9) we find, since $\delta_k = 0$ for $1 \leqslant k < K$, that

$$\alpha = x\prod_{k=K}^{\infty}\left(1 - \frac{\delta_k}{n_k}\right) \geqslant x\prod_{k=K}^{\infty}\left(1 + \frac{1}{n_k}\log\left(1-2\left(\frac{1+x}{2}\right)^{-n_k}\right)\right)$$
$$\geqslant x\prod_{k=K}^{\infty}\left(1+\frac{x+1}{x-1}2^k\right)\left(2+\frac{x+1}{x-1}2^k\right)^{-1}$$
$$\geqslant x\prod_{k=K}^{\infty}(1-2^{-k}) > x\left(1-\frac{\varepsilon}{x}\right) = x - \varepsilon$$

by (10) and (4). Hence (1) is proved.

It only remains to verify (2). We shall first prove that

$$(11) \qquad x_m^{n_k} \geqslant a_k + \tfrac{1}{2}\eta_k + 2^{-mn_k} \quad \text{for} \quad m \geqslant k > 0.$$

Equality obviously holds for $m = k$, by (7); suppose therefore that the inequality is true for $m-1$. We then have for $m > k$

$$x_m^{n_k} = x_{m-1}^{n_k}\left(1-\frac{\delta_m}{n_m}\right)^{n_k} \qquad \text{(by (8))}$$

$$= (a_k+\tfrac{1}{2}\eta_k+2^{-(m-1)n_k})\left(1+\frac{1}{n_m}\log\left(1-2\left(\frac{1+x}{2}\right)^{-n_m}\right)\right)^{n_k}$$

(by (9) and (11))

$$\geqslant (a_k+\tfrac{1}{2}\eta_k+2^{-(m-1)n_k})(1-2^{-mn_k}x^{-n_k}) \qquad \text{(by (3))}$$

$$\geqslant (a_k+\tfrac{1}{2}\eta_k+2^{-(m-1)n_k})(1-2^{-mn_k}x_{k-1}^{-n_k}) \qquad \text{(by (5))}$$

$$\geqslant a_k+\tfrac{1}{2}\eta_k+2^{-(m-1)n_k}-2^{-mn_k} \qquad \text{(by (6))}$$

$$\geqslant a_k+\tfrac{1}{2}\eta_k+2^{-mn_k},$$

and (11) is proved. But (11) implies

$$\alpha^{n_k} = \lim_{m\to\infty} x_m^{n_k} \geqslant a_k+\tfrac{1}{2}\eta_k.$$

On the other hand

$$\alpha^{n_k} \leqslant x_k^{n_k} = a_k+\tfrac{1}{2}\eta_k+2^{-kn_k}$$

by (7), therefore

$$|\alpha^{n_k}-a_k|^{1/n_k} \leqslant 2^{-k} \quad \text{if} \quad \eta_k = 0,$$

$$2^{-1/n_k} \leqslant |\alpha^{n_k}-a_k|^{1/n_k} \leqslant (\tfrac{1}{2}+2^{-kn_k})^{1/n_k} \quad \text{if} \quad \eta_k = 1,$$

and (2) holds with $g_{n_k} = a_k$.

Proof of (c). Let ε, a and b be real numbers satisfying

$$0 \leqslant \varepsilon \leqslant 1 \leqslant a < b,$$

and denote by $A(\varepsilon)$ the set of all $\alpha \in A$ satisfying $P(\alpha) \leqslant 1-\varepsilon$, and by $A(\varepsilon, a, b)$ the subset of those $\alpha \in A(\varepsilon)$ for which

$$a \leqslant \alpha \leqslant b.$$

The upper bound for $P(\alpha)$ means that there exists to α an infinite set N of positive integers n satisfying

$$(12) \qquad |\alpha^n-g_n| \leqslant (1-\tfrac{1}{2}\varepsilon)^n, \qquad g_n \geqslant 2.$$

Therefore, if $\alpha \in A(\varepsilon, a, b)$, then for each such n,

$$\tfrac{1}{2}a^n \leqslant \tfrac{1}{2}\alpha^n \leqslant g_n \leqslant 2\alpha^n \leqslant 2b^n,$$

because

$$(1-\tfrac{1}{2}\varepsilon)^n \leqslant 1 \leqslant \tfrac{1}{2}g_n.$$

This further implies that, if $n \in N$ is given, the integer g_n has not more than $2b^n$ possibilities. Next, if both $n \in N$ and g_n are given, then a is by (12) restricted to the interval

$$I_n(g_n):\ \{g_n - (1-\tfrac{1}{2}\varepsilon)^n\}^{1/n} \leqslant a \leqslant \{g_n + (1-\tfrac{1}{2}\varepsilon)^n\}^{1/n}$$

of length

$$\{g_n + (1-\tfrac{1}{2}\varepsilon)^n\}^{1/n} - \{g_n - (1-\tfrac{1}{2}\varepsilon)^n\}^{1/n} \sim \frac{2(1-\frac{1}{2}\varepsilon)^n}{n g_n^{(n-1)/n}}.$$

For large n this is less than $(1-\frac{1}{2}\varepsilon)^n / a^{n-1}$ because

$$g_n^{(n-1)/n} \geqslant 2^{-(n-1)/n} a^{n-1} > \frac{2}{n} a^{n-1}.$$

Therefore, for each sufficiently large element n of N, the total length of all the intervals $I_n(g_n)$ corresponding to possible values of g_n is less than

$$2b^n \frac{(1-\frac{1}{2}\varepsilon)^n}{a^{n-1}} = 2a\left(\frac{(1-\frac{1}{2}\varepsilon)b}{a}\right)^n.$$

This again implies that every point a of $A(\varepsilon, a, b)$ lies in the union of a countable set of intervals of total length not exceeding

$$S_m = \sum_{n=m}^{\infty} 2a\left(\frac{(1-\frac{1}{2}\varepsilon)b}{a}\right)^n,$$

where m can be chosen as large as we please.

If now

$$b < \frac{a}{1-\frac{1}{2}\varepsilon},$$

then S_1 converges, and hence $A(\varepsilon, a, b)$ has the Lebesgue measure zero.

Since the set $A(\varepsilon)$ can be written as

$$A(\varepsilon) = \bigcup_{n=1}^{\infty} A\left(\varepsilon, (1-\tfrac{1}{3}\varepsilon)^{-(n-1)}, (1-\tfrac{1}{3}\varepsilon)^{-n}\right),$$

it evidently is a union of countably many sets all of measure zero. Therefore $A(\varepsilon)$ and hence also $\bigcup_{n=1}^{\infty} A(1/n)$ have the measure zero, which proves the assertion.

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