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Applications of Some Formulae by Hermite to the Approximation of Exponentials and Logarithms

To C. L. SIEGEL on his 70th birthday

K. MAHLER

While LIOUVILLE gave the first examples of transcendental numbers, the modern theory of proofs of transcendency started with Hermite's beautiful paper "Sur la fonction exponentielle" (HERMITE, 1873). In this paper, for a given system of distinct complex numbers $\omega_0, \omega_1, \ldots, \omega_m$ and of positive integers $\varrho_0, \varrho_1, \ldots, \varrho_m$ with the sum σ, HERMITE constructed a set of $m+1$ polynomials

$$\mathfrak{A}_0(z), \mathfrak{A}_1(z), \ldots, \mathfrak{A}_m(z)$$

of degrees not exceeding $\sigma-\varrho_0, \sigma-\varrho_1, \ldots, \sigma-\varrho_m$, respectively, such that all the functions

$$\mathfrak{A}_k(z)\, e^{\omega_l z} - \mathfrak{A}_l(z)\, e^{\omega_k z} \qquad (0 \leqq k < l \leqq m)$$

vanish at $z=0$ at least to the order $\sigma+1$. On putting $z=1$, these formulae produce simultaneous rational approximations of the numbers $1, e, e^2, \ldots, e^m$ that are so good that they imply the linear independence of these numbers and hence the transcendency of e.

In a later paper (HERMITE, 1893), Hermite introduced a second system of polynomials

$$A_0(z), A_1(z), \ldots, A_m(z)$$

of degrees at most $\varrho_0-1, \varrho_1-1, \ldots, \varrho_m-1$, respectively, for which the sum

$$\sum_{k=0}^{m} A_k(z)\, e^{\omega_k z}$$

vanishes at $z=0$ at least to the order $\sigma-1$. On putting again $z=1$, one obtains now a linear form

$$a_0 + a_1 e + \cdots + a_m e^m$$

of small absolute value and with small integral coefficients, from which again the transcendency of e may be deduced. Surprisingly, HERMITE himself never took this step, and I was seemingly the first to use the polynomials $A_k(z)$ for this purpose (MAHLER, 1931).

In the present paper I once more wish to exhibit the usefulness of Hermite's polynomials $A_k(z)$ for the study of transcendental numbers. I shall prove a number of explicit estimates, free from any unknown constants, for the simultaneous rational approximations of powers of e or of the natural logarithms of sets of rational numbers.

1. Let $\omega_0, \omega_1, \ldots, \omega_m, \Omega$ be $m+2$ integers satisfying

$$0=\omega_0<\omega_1<\omega_2<\cdots<\omega_m=\Omega,$$

and let

$$M_k=\left|\prod_{\substack{l=0\\ l\neq k}}^{m}(\omega_k-\omega_l)\right|, \quad M=\operatorname*{lcm}_{k=0,1,\ldots,m} M_k, \quad N=\operatorname*{lcm}_{\substack{l\neq k\\ k,l=0,1,\ldots,m}}(\omega_k-\omega_l),$$

where lcm denotes the least common multiple. Let z be any complex number, ϱ a positive integer, and

$$\delta_{hk}=\begin{cases}1 & \text{if} \quad h=k,\\ 0 & \text{if} \quad h\neq k,\end{cases}$$

the Kronecker sign. Denote by C_0 and C_∞ two circles in the complex $\mathfrak{z}$-plane, both with centres at $\mathfrak{z}=0$, and of radii less than 1, and greater than Ω, respectively. Then put

$$A_{hk}(z)=\frac{1}{2\pi i}\int\limits_{C_0}\frac{e^{z\mathfrak{z}}\,d\mathfrak{z}}{\prod\limits_{l=0}^{m}(\mathfrak{z}+\omega_k-\omega_l)^{\varrho+\delta_{hl}}}, \quad R_h(z)=\frac{1}{2\pi i}\int\limits_{C_\infty}\frac{e^{z\mathfrak{z}}\,d\mathfrak{z}}{\prod\limits_{l=0}^{m}(\mathfrak{z}-\omega_l)^{\varrho+\delta_{hl}}}.$$

These definitions imply (see, e.g. MAHLER, 1931) that $A_{hk}(z)$ is a polynomial in z at most of degree ϱ; that

$$R_h(z)=\sum_{k=0}^{m}A_{hk}(z)\,r^{\omega_k z} \qquad (h=0, 1, \ldots, m),$$

and that the determinant

$$D(z)=\begin{vmatrix}A_{00}(z), & \ldots, & A_{0m}(z)\\ \vdots & & \vdots\\ A_{m0}(z), & \ldots, & A_{mm}(z)\end{vmatrix}=Cz^{(m+1)\varrho},$$

where $C\neq 0$ does not depend on z.

2. By the paper quoted, $R_h(z)$ may also be written as

$$R_h(z)=z^{(m+1)\varrho}\int_0^1 dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{m-1}}dt_m\,\Phi(t)\,e^{z\Psi(t)},$$

where the expressions Φ and Ψ are defined by

$$\Phi(t)=\frac{(1-t_1)^{\varrho+\delta_{h0}-1}(t_1-t_2)^{\varrho+\delta_{h1}-1}\ldots(t_{m-1}-t_m)^{\varrho+\delta_{h,m-1}-1}\,t_m^{\varrho+\delta_{hm}-1}}{\prod\limits_{l=0}^{m}(\varrho+\delta_{hl}-1)!}$$

and

$$\Psi(t)=\omega_0(1-t_1)+\omega_1(t_1-t_2)+\cdots+\omega_{m-1}(t_{m-1}-t_m)+\omega_m t_m,$$

respectively. Here the quantities

$$1-t_1, t_1-t_2, \ldots, t_{m-1}-t_m, t_m$$

are non-negative and have the sum 1. Therefore, by the theorem on the arithmetic and geometric means,

$$0 \leqq (1-t_1)(t_1-t_2)\dots(t_{m-1}-t_m)t_m \leqq (m+1)^{-(m+1)},$$

so that

$$0 \leqq \Phi(t) \leqq (m+1)^{-(m+1)}(\varrho!(\varrho-1)!^m)^{-1}.$$

Further

$$0 \leqq \Psi(t) \leqq \Omega$$

and

$$\int_0^1 dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{m-1}} dt_m = \frac{1}{m!}.$$

It follows then from the first mean value theorem that

$$|R_h(z)| \leqq \frac{|z|^{(m+1)\varrho}\, e^{\Omega|z|}}{m!\,(m+1)^{(m+1)(\varrho-1)}\,\varrho!\,(\varrho-1)!^m}.$$

3. From the integral, $A_{hk}(z)$ is the polynomial

$$A_{hk}(z) = \sum_{j=0}^{\varrho} A_{hk}^{(j)} \frac{z^j}{j!}$$

where the general coefficient $A_{hk}^{(j)}$ is given by

$$A_{hk}^{(j)} = \frac{1}{2\pi i} \int_{C_0} \frac{\mathfrak{z}^j\, d\mathfrak{z}}{\prod_{l=0}^{m} (\mathfrak{z}+\omega_k-\omega_l)^{\varrho+\delta_{hl}}} \qquad (j=0, 1, \dots, \varrho).$$

If we choose for C_0 the circle

$$|\mathfrak{z}| = \frac{1}{m+1},$$

then on this circle,

$$\left|1+\frac{\mathfrak{z}}{\omega_k-\omega_l}\right| \geqq 1-|\mathfrak{z}| = 1-\frac{1}{m+1} = \frac{m}{m+1} \quad \text{for} \quad k \neq l.$$

The formula for $A_{hk}^{(j)}$ may also be written as

$$A_{hk}^{(j)} = \prod_{\substack{l=0\\ l\neq k}}^{m} (\omega_k-\omega_l)^{-\varrho-\delta_{hl}} \cdot \frac{1}{2\pi i} \int_{C_0} \frac{\mathfrak{z}^{j-\varrho-\delta_{hk}}\, d\mathfrak{z}}{\prod_{\substack{l=0\\ l\neq k}}^{m} \left(1+\frac{\mathfrak{z}}{\omega_k-\omega_l}\right)^{\varrho+\delta_{hl}}}.$$

It follows therefore that

$$|A_{hk}^{(j)}| \leqq M_k^{-\varrho} \cdot \frac{1}{2\pi} \frac{2\pi}{m+1} \left(\frac{1}{m+1}\right)^{-(\varrho+\delta_{hk})} \left(\frac{m}{m+1}\right)^{-m\varrho-(1-\delta_{hk})},$$

and so, by $0 \leqq \delta_{hk} \leqq 1$, that

$$|A_{hk}^{(j)}| \leqq M_k^{-\varrho} m^{-m\varrho}(m+1)^{(m+1)\varrho}.$$

4. From the original integral,

$$A_{hk}(z) = \prod_{\substack{l=0\\ l\neq k}}^{m} \left(\omega_k - \omega_l + \frac{d}{dz}\right)^{-\varrho-\delta_{hl}} \frac{z^{\varrho+\delta_{hk}-1}}{(\varrho+\delta_{hk}-1)!}.$$

This formula may also be written as

$$A_{hk}(z) = \prod_{\substack{l=0\\ l\neq k}}^{m} (\omega_k - \omega_l)^{-\varrho-\delta_{hl}} \cdot \prod_{\substack{l=0\\ l\neq k}}^{m} \left(1 + \frac{1}{\omega_k - \omega_l}\,\frac{d}{dz}\right)^{-\varrho-\delta_{hl}} \frac{z^{\varrho+\delta_{hk}-1}}{(\varrho+\delta_{hk}-1)!},$$

or, what is the same,

$$A_{hk}(z) = \prod_{\substack{l=0\\ l\neq k}}^{m} (\omega_k - \omega_l)^{-\varrho-\delta_{hl}} \cdot \prod_{\substack{l=0\\ l\neq k}}^{m} \left\{\sum_{\lambda=0}^{\infty} \binom{-\varrho-\delta_{hl}}{\lambda} (\omega_k - \omega_l)^{-\lambda} \frac{d^\lambda}{dz^\lambda}\right\} \frac{z^{\varrho+\delta_{hk}-1}}{(\varrho+\delta_{hk}-1)!}.$$

Here the binomial coefficients are integers; the differences $\omega_k - \omega_l$ are divisors of N; and hence the operator has the form

$$\prod_{\substack{l=0\\ l\neq k}}^{m} \left\{\sum_{\lambda=0}^{\infty} \binom{-\varrho-\delta_{hl}}{\lambda} (\omega_k - \omega_l)^{-\lambda} \frac{d^\lambda}{dz^\lambda}\right\} = \sum_{\lambda=0}^{\infty} g_\lambda N^{-\lambda} \frac{d^\lambda}{dz^\lambda}$$

where

$$g_0, g_1, g_2, \ldots \qquad (g_0 = 1)$$

are certain integers that also depend on h and k. It follows that

$$A_{hk}(z) = \prod_{\substack{l=0\\ l\neq k}}^{m} (\omega_k - \omega_l)^{-\varrho+\delta_{hl}} \cdot \sum_{\lambda=0}^{\varrho+\delta_{hk}-1} g_\lambda N^{-\lambda} \frac{z^{\varrho+\delta_{hk}-\lambda-1}}{(\varrho+\delta_{hk}-\lambda-1)!}.$$

Here, from the definitions of M and N, the factor

$$M^\varrho N \cdot \prod_{\substack{l=0\\ l\neq k}}^{m} (\omega_k - \omega_l)^{-\varrho-\delta_{hl}}$$

is an integer. Therefore the product

$$a_{hk}(z) = M^\varrho N^{\varrho+1} \varrho!\, A_{hk}(z), = \sum_{j=0}^{\varrho} a_{hk}^{(j)} z^j$$

say, is a polynomial in z with integral coefficients $a_{hk}^{(j)}$.

Since

$$a_{hk}(z) = M^\varrho N^{\varrho+1} \varrho! \sum_{j=0}^{\varrho} A_{hk}^{(j)} \frac{z^j}{j!},$$

these integral coefficients can be written in the form

$$a_{hk}^{(j)} = M^\varrho N^{\varrho+1} \varrho!\, \frac{A_{hk}^{(j)}}{j!}$$

and so satisfy the inequality

$$|a_{hk}^{(j)}| \leqq \frac{M^{\varrho} N^{\varrho+1} \varrho!\,(m+1)^{(m+1)\varrho}}{M_k^{\varrho}\, m^{m\varrho}}.$$

It is further obvious that

$$|a_{hk}(z)| \leqq \frac{M^{\varrho} N^{\varrho+1} \varrho!\,(m+1)^{(m+1)\varrho}\, e^{|z|}}{M_k^{\varrho}\, m^{m\varrho}}$$

because

$$\sum_{j=0}^{\varrho} \frac{|z|^j}{j!} < e^{|z|}.$$

In analogy to $a_{hk}(z)$ put also

$$r_h(z) = M^{\varrho} N^{\varrho+1} \varrho!\, R_h(z) \qquad (h=0, 1, \ldots, m).$$

Then

$$r_h(z) = \sum_{k=0}^{m} a_{hk}(z)\, e^{\omega_k z} \qquad (h=0, 1, \ldots, m).$$

From the identity for $D(z)$, the new determinant

$$d(z) = \begin{vmatrix} a_{00}(z), & \ldots, & a_{0m}(z) \\ \vdots & & \vdots \\ a_{m0}(z), & \ldots, & a_{mm}(z) \end{vmatrix} = c\, z^{(m+1)\varrho}$$

where again $c \neq 0$ is independent of z.

We note that, by the estimate for $R_h(z)$,

$$|r_h(z)| \leqq \frac{M^{\varrho} N^{\varrho+1} |z|^{(m+1)\varrho}\, e^{\Omega|z|}}{m!\,(m+1)^{(m+1)(\varrho-1)}\,(\varrho-1)!^m}.$$

5. The inequalities just proved can be simplified by means of some simple lower and upper bounds for M_k, M, and N.

First, the factors of M_k are integers distinct from one another and from zero, and of these factors k are positive and $m-k$ are negative. It follows therefore at once that

$$M_k \geqq k!\,(m-k)! = m! \binom{m}{k}^{-1} \geqq 2^{-m} m!$$

Secondly, N is the least common multiple of certain positive integers not greater than Ω, and hence

$$N \leqq \operatorname{lcm}(1, 2, \ldots, \Omega) \leqq e^{1.04\,\Omega}$$

where the numerical inequality is taken from the paper (ROSSER and SCHOENFELD, 1962).

Thirdly, an upper bound for M may be obtained by the following method due to B. H. NEUMANN.

For each suffix k and for each prime p let $\mu_k(p)$ denote the largest integer for which

$$p^{\mu_k(p)} | M_k.$$

Hence

$$M_k = \prod_p p^{\mu_k(p)}.$$

Since $|\omega_k - \omega_l| \leqq \Omega$, a power p^t of p cannot be a divisor of some factor $\omega_k - \omega_l$ of M_k unless

$$p^t \leqq \Omega \quad \text{and therefore} \quad p \leqq \Omega.$$

The largest possible value of t is then

$$\tau = \left[\frac{\log \Omega}{\log 2}\right],$$

because $2^{\tau+1} > \Omega$.

One counts as usual how many of the factors

$$\omega_k - \omega_l, \quad \text{where} \quad 0 \leqq l \leqq m, \quad l \neq k,$$

are successively divisible by p^1, by p^2, by p^3, etc., and finally by p^τ; the sum of all these numbers is equal to $\mu_k(p)$. Now M_k has just m factors $\omega_k - \omega_l$, and so none of these numbers can exceed m. Also these factors of M_k lie in the interval from $\omega_k - \Omega$ to ω_k of length Ω, and this interval contains the multiple 0 of p^t which is not a factor of M_k. Therefore at most

$$\min\left(m, \left[\frac{\Omega}{p^t}\right]\right)$$

factors of M_k are divisible by p^t, whence

$$\mu_k(p) \leqq \sum_{t=1}^{\tau} \min\left(m, \left[\frac{\Omega}{p^t}\right]\right).$$

We replace this inequality by the weaker but more convenient one,

$$\mu_k(p) \leqq \min\left(m, \left[\frac{\Omega}{p}\right]\right) + \sum_{t=2}^{\tau} \left[\frac{\Omega}{p^t}\right], = \mu(p) \text{ say}.$$

Let

$$M^* = \prod_{p \leqq \Omega} p^{\mu(p)}.$$

Then all products M_k and so also their least common multiple M are divisors of M^*, and hence it follows that

$$M \leqq M^*.$$

6. Put now

$$\nu(p) = \sum_{t=1}^{\tau} \left[\frac{\Omega}{p^t}\right],$$

so that, by a well known formula,

$$\Omega! = \prod_{p \leqq \Omega} p^{\nu(p)}.$$

It follows that

$$M^* = \frac{\Omega!}{\Lambda}$$

where Λ denotes the product

$$\Lambda = \prod_{p \leqq \Omega} p^{\nu(p)-\mu(p)}.$$

From the definitions of $\mu(p)$ and $\nu(p)$,

$$\nu(p)-\mu(p) = \begin{cases} \left[\dfrac{\Omega}{p}\right] - m & \text{if} \quad p \leqq \dfrac{\Omega}{m}, \\ 0 & \text{if} \quad p > \dfrac{\Omega}{m}, \end{cases}$$

so that

$$\Lambda = \prod_{p \leqq \frac{\Omega}{m}} p^{\left[\frac{\Omega}{p}\right]-m}$$

and therefore also

$$\log \Lambda \geqq \sum_{p \leqq \frac{\Omega}{m}} \left(\Omega \frac{\log p}{p} - (m+1)\log p\right).$$

In the paper (ROSSER and SCHOENFELD, 1962), it is proved that

$$\sum_{p \leqq x} \frac{\log p}{p} > \log x + E - \frac{1}{2\log x} \quad \text{for} \quad x > 1$$

and

$$\sum_{p \leqq x} \log p < 1.02 x \quad \text{for} \quad x \geqq 1,$$

where E is a certain constant satisfying

$$E > -1.34.$$

Assume for the moment that

$$\Omega \geqq e^2 m > m,$$

and therefore

$$2\log\frac{\Omega}{m} \geqq 4, \qquad \frac{1}{2\log\frac{\Omega}{m}} \leqq 0.25,$$

while trivially

$$\frac{m+1}{m} \leqq 2.$$

It follows then that

$$\log \Lambda > \Omega\left(\log\frac{\Omega}{m} - 1.34 - \frac{1}{2\log\frac{\Omega}{m}}\right) - (m+1)\,1.02\,\frac{\Omega}{m}$$

or

$$\log \Lambda > \Omega\left\{\log \frac{\Omega}{m} - \left(1.34 + \frac{1}{2\log \frac{\Omega}{m}} + \frac{m+1}{m} 1.02\right)\right\},$$

and here

$$1.34 + \frac{1}{2\log \frac{\Omega}{m}} + \frac{m+1}{m} 1.02 \leqq 1.34 + 0.25 + 2.04 = 3.63 < \frac{11}{3}.$$

Hence, finally,

$$\log \Lambda > \Omega\left(\log \frac{\Omega}{m} - \frac{11}{3}\right),$$

that is,

$$\Lambda > \left(\frac{\Omega}{m}\right)^{\Omega} e^{-\frac{11}{3}\Omega}.$$

This inequality trivially is valid also for

$$\Omega < e^2 m,$$

because then

$$\Lambda \geqq 1 > e^{\left(2-\frac{11}{3}\right)\Omega} > \left(\frac{\Omega}{m}\right)^{\Omega} e^{-\frac{11}{3}\Omega}.$$

7. Thus it has been proved that always

$$M \leqq M^* \leqq \frac{\Omega!}{\Lambda} < \Omega! \left(\frac{\Omega}{m}\right)^{-\Omega} e^{\frac{11}{3}\Omega}.$$

Here

$$\Omega! < e\sqrt{\Omega}\,\Omega^{\Omega} e^{-\Omega}$$

and therefore

$$M < e\sqrt{\Omega}\, m^{\Omega} e^{\frac{8}{3}\Omega}.$$

But $\Omega \geqq 1$, hence

$$e\sqrt{\Omega} = e^{1+\frac{1}{2}\log\{1+(\Omega-1)\}} \leqq e^{\frac{1}{2}(\Omega+1)+\frac{1}{2}(\Omega-1)} = e^{\Omega},$$

and so finally

$$M < m^{\Omega} e^{\frac{11}{3}\Omega}.$$

On combining this inequality with the earlier one for N,

$$M^{\varrho} N^{\varrho+1} \leqq M^{\varrho} N^{2\varrho} < (m^{\Omega} e^{\frac{11}{3}\Omega})^{\varrho} (e^{1.04\Omega})^{2\varrho}$$

and hence

$$M^{\varrho} N^{\varrho+1} < m^{\Omega\varrho} e^{6\Omega\varrho}.$$

8. For the moment put

$$a = \frac{M^{\varrho} N^{\varrho+1} \varrho! (m+1)^{(m+1)\varrho}}{M_k^{\varrho} m^{m\varrho}}, \quad r = \frac{M^{\varrho} N^{\varrho+1}}{m! (m+1)^{(m+1)(\varrho-1)} (\varrho-1)!^m};$$

by what has been proved in § 4,

$$\max_{h,k,j} |a_{hk}^{(j)}| \leqq a, \quad \max_{h,k} |a_{hk}(z)| \leqq a\, e^{|z|}, \quad \max_{h} |r_h(z)| \leqq r |z|^{(m+1)\varrho} e^{\Omega |z|}.$$

Thus upper bounds for a and r imply upper bounds for $|a_{hk}^{(j)}|$, $|a_{hk}(z)|$, and $|r_h(z)|$. Such upper bounds are obtained as follows.

To begin with a, we apply in addition to

$$M_k \geqq 2^{-m} m! \quad \text{and} \quad M^{\varrho} N^{\varrho+1} < m^{\Omega\varrho} e^{6\Omega\varrho}$$

the formulae

$$\sqrt{2\pi\varrho}\, \varrho^{\varrho} e^{-\varrho} < \varrho! < e \sqrt{\varrho}\, \varrho^{\varrho} e^{-\varrho}, \quad m! > \sqrt{2\pi m}\, m^m e^{-m}.$$

We find then that

$$a < \frac{m^{\Omega\varrho} e^{6\Omega\varrho} \cdot e \sqrt{\varrho}\, \varrho^{\varrho} e^{-\varrho} \cdot (m+1)^{(m+1)\varrho}}{(2^{-m} \cdot \sqrt{2\pi m}\, m^m e^{-m})^{\varrho} m^{m\varrho}}$$

$$= \left(\frac{e^2 \varrho}{(2\pi)^{\varrho}} \right)^{1/2} \left(\frac{2^m e^{m-1} (m+1)^{m+1}}{m^{2m+\frac{1}{2}}} \cdot \varrho\, m^{\Omega} e^{6\Omega} \right)^{\varrho}.$$

Here

$$e^2 < 7.5, \quad 2\pi > 6$$

and hence

$$\frac{e^2 \varrho}{(2\pi)^{\varrho}} < \frac{7.5\varrho}{(1+5)^{\varrho}} \leqq \frac{7.5\varrho}{1+5\varrho} < \frac{3}{2} < 4.$$

Further the function

$$\frac{2^m e^{m-1} (m+1)^{m+1}}{m^{2m+\frac{1}{2}}}$$

of m assumes it maximum when $m = 2$, and this maximum has the value

$$\frac{27e}{\sqrt{32}} < 13.$$

The final result is therefore

$$a < 2(13\varrho\, m^{\Omega} e^{6\Omega})^{\varrho}$$

and it follows that

$$\boxed{\max_{h,k,j} |a_{hk}^{(j)}| < 2(13\varrho\, m^{\Omega} e^{6\Omega})^{\varrho}, \quad \max_{h,k} |a_{hk}(z)| < 2(13\varrho\, m^{\Omega} e^{6\Omega})^{\varrho} e^{|z|}.}$$

8. Since

$$(\varrho-1)! > \sqrt{\frac{2\pi}{\varrho}}\, \varrho^{\varrho} e^{-\varrho},$$

we similarly find that

$$r<\frac{m^{\Omega\varrho}\,e^{6\Omega\varrho}}{\sqrt{2\pi m}\,m^m\,e^{-m}\cdot(m+1)^{(m+1)(\varrho-1)}\cdot\left(\sqrt{\frac{2\pi}{\varrho}}\,\varrho^\varrho\,e^{-\varrho}\right)^m}$$

$$=\frac{e^m\,\varrho^{\frac{m}{2}}\,m^{m\varrho}}{(2\pi)^{\frac{m+1}{2}}\cdot m^m(m+1)^{(m+1)(\varrho-1)}\cdot e^{m\varrho}}\left(\frac{m^\Omega\,e^{6\Omega}\,e^{2m}}{m^m\,\varrho^m}\right)^\varrho.$$

Here $m\geqq 1$ and $\varrho\geqq 1$. Further

$$m^m(m+1)^{(m+1)(\varrho-1)}=m^{m\varrho+\varrho-1}\left(1+\frac{1}{m}\right)^{(m+1)(\varrho-1)}\geqq m^{m\varrho+\varrho-1}\,e^{\varrho-1}\geqq m^{m\varrho}$$

because

$$\left(1+\frac{1}{m}\right)^{m+1}>e\,;$$

and also

$$(2\pi)^{\frac{m+1}{2}}>1\,.$$

It follows that

$$\frac{e^m\,\varrho^{\frac{m}{2}}\,m^{m\varrho}}{(2\pi)^{\frac{m+1}{2}}\,m^m(m+1)^{(m+1)(\varrho-1)}\,e^{m\varrho}}<\left(\frac{\varrho^{\frac{1}{2}}}{e^{\varrho-1}}\right)^m\leqq 1\,,$$

since

$$e^{\varrho-1}\geqq 1+(\varrho-1)=\varrho\geqq\varrho^{\frac{1}{2}}\,.$$

The final result is then that

$$r<\left(\frac{m^\Omega\,e^{6\Omega}\,e^{2m}}{m^m\,\varrho^m}\right)^\varrho\leqq\left(\frac{m^\Omega\,e^{8\Omega}}{m^m\,\varrho^m}\right)^\varrho\,;$$

here we have used that

$$m\leqq\Omega\quad\text{and hence}\quad e^m\leqq e^\Omega\,.$$

Thus it has been established that

$$\boxed{\max_h|r_h(z)|<\left(\frac{m^\Omega\,e^{8\Omega}}{m^m\,\varrho^m}\right)^\varrho|z|^{(m+1)\varrho}\,e^{\Omega|z|}}$$

9. As a first application, denote by ω a positive integer and put

$$z=\frac{1}{\omega}.$$

Let further $q\geqq 1,\ q_1, q_2, \ldots, q_m$ be $m+1$ arbitrary integers, and let

$$\varepsilon=2mq\max_{k=1,2,\ldots,m}\left|e^{\frac{\omega_k}{\omega}}-\frac{q_k}{q}\right|$$

and

$$\varepsilon_k = 2mq\left(e^{\frac{\omega_k}{\omega}} - \frac{q_k}{q}\right) \qquad (k=0, 1, \ldots, m),$$

where we have put

$$q_0 = q \geqq 1 .$$

Since $\omega_0 = 0$, trivially

$$\varepsilon_0 = 0 ,$$

and hence

$$\varepsilon = \max_{k=0,1,\ldots,m} |\varepsilon_k| = \max_{k=1,2,\ldots,m} |\varepsilon_k| .$$

The powers

$$e^{\frac{\omega_1}{\omega}}, e^{\frac{\omega_2}{\omega}}, \ldots, e^{\frac{\omega_m}{\omega}}$$

are irrational numbers, and hence

$$\varepsilon > 0 .$$

We shall now establish a positive lower estimate for ε.

For this purpose we note that the $(m+1)^2$ numbers

$$\omega^{\varrho} a_{hk}\left(\frac{1}{\omega}\right), \quad = A_{hk} \text{ say} \qquad (h, k=0, 1, \ldots, m)$$

are integers, with the determinant

$$\begin{vmatrix} A_{00}, & \ldots, & A_{0m} \\ \vdots & & \vdots \\ A_{m0}, & \ldots, & A_{mm} \end{vmatrix} \neq 0 .$$

On putting

$$\omega^{\varrho} r_h\left(\frac{1}{\omega}\right) = R_h \qquad (h=0, 1, \ldots, m),$$

we have

$$R_h = \sum_{k=0}^{m} A_{hk} e^{\frac{\omega_k}{\omega}} \qquad (h=0, 1, \ldots, m).$$

The estimates in §§ 7—8 now take the form

$$\max_{h,k} |A_{hk}| < 2(13\varrho\omega m^{\Omega} e^{6\Omega})^{\varrho} e^{\frac{1}{\omega}}$$

and

$$\max_{h} |R_h| < \left(\frac{m^{\Omega} e^{8\Omega}}{m^m \varrho^m \omega^m}\right)^{\varrho} e^{\frac{\Omega}{\omega}} .$$

10. Since the determinant of the integers A_{hk} is distinct from zero, and since the integers

$$q_0 \geqq 1, \quad q_1, \ldots, q_m$$

do not all vanish, there exists a suffix h such that

$$\sum_{k=0}^{m} A_{hk} q_k \neq 0$$

and that therefore

$$\left| \sum_{k=0}^{m} A_{hk} q_k \right| \geqq 1 .$$

With this value of h, put

$$Q = \frac{1}{q} \sum_{k=0}^{m} A_{hk} q_k , \quad E = \frac{1}{2mq} \sum_{k=1}^{m} A_{hk} \varepsilon_k .$$

From the definition of ε_k,

$$R_h = \sum_{k=0}^{m} A_{hk} e^{\frac{\omega_k}{\omega}} = \sum_{k=0}^{m} A_{hk} \left(\frac{q_k}{q} + \frac{\varepsilon_k}{2mq} \right) = Q + E .$$

Here

$$|Q| \geqq \frac{1}{q}$$

and

$$|E| \leqq \frac{\varepsilon}{2q} \max_{h,k} |A_{hk}| .$$

Assume now that

$$\max_{h} |R_h| \leqq \frac{1}{2q} .$$

It follows then that

$$|E| \geqq \frac{1}{2q}$$

and hence that

$$\varepsilon \max_{h,k} |A_{hk}| \geqq 1 .$$

Thus the following result is obtained.

If

$$\left(\frac{m^{\Omega} e^{8\Omega}}{m^m \varrho^m \omega^m} \right)^{\varrho} e^{\frac{\Omega}{\omega}} \leqq \frac{1}{2q} ,$$

then

$$\varepsilon > \{ 2 e^{\frac{1}{\omega}} (13 \varrho \omega m^{\Omega} e^{6\Omega})^{\varrho} \}^{-1} .$$

This result can be slightly simplified. Since all three integers ω, Ω, ϱ are at least 1,

$$2e^{\frac{\Omega}{\omega}} \leqq 2e^{\Omega} < e^{2\Omega} \leqq e^{2\Omega\varrho},$$

so that

$$2\left(\frac{m^{\Omega} e^{8\Omega}}{m^m \varrho^m \omega^m}\right)^{\varrho} e^{\frac{\Omega}{\omega}} < \left(\frac{m^{\Omega} e^{10\Omega}}{m^m \varrho^m \omega^m}\right)^{\varrho}.$$

Further

$$\frac{\varepsilon}{2mq} > \{4e^{\frac{1}{\omega}} m(13\varrho\omega m^{\Omega} e^{6\Omega})^{\varrho}\}^{-1} q^{-1} > (52e\varrho\omega m^{\Omega+1} e^{6\Omega})^{-\varrho} q^{-1}.$$

Here

$$52 < e^4,$$

and so

$$\frac{\varepsilon}{2mq} > (\varrho\omega m^{\Omega+1} e^{6\Omega+5})^{-\varrho} q^{-1}.$$

Thus the following result holds.

Lemma 1. *If ϱ is chosen such that*

$$\left(\frac{m^{\Omega} e^{10\Omega}}{m^m \varrho^m \omega^m}\right)^{\varrho} \leqq \frac{1}{q},$$

then

$$\max_{k=1,2,\ldots,m} \left| e^{\frac{\omega_k}{\omega}} - \frac{q_k}{q} \right| > (\varrho\omega m^{\Omega+1} e^{6\Omega+5})^{-\varrho} q^{-1}.$$

11. When applying this lemma, one naturally will choose the integer ϱ as small as possible because this improves the estimate. It is now convenient to distinguish between the two cases $\varrho = 1$ and $\varrho > 1$.

The case $\varrho = 1$ holds exactly when

$$\omega \geqq (m^{\Omega-m} e^{10\Omega} q)^{\frac{1}{m}},$$

and then, by the lemma,

$$\max_{k=1,2,\ldots,m} \left| e^{\frac{\omega_k}{\omega}} - \frac{q_k}{q} \right| > (\omega m^{\Omega+1} e^{6\Omega+5})^{-1} q^{-1}.$$

Next, excluding this case, let

$$\omega < (m^{\Omega-m} e^{10\Omega} q)^{\frac{1}{m}},$$

so that the smallest possible value for ϱ is at least 2. This value ϱ satisfies the inequality

$$\left(\frac{m^{\Omega} e^{10\Omega}}{m^m \varrho^m \omega^m}\right)^{\varrho} \leqq \frac{1}{q} < \left(\frac{m^{\Omega} e^{10\Omega}}{m^m (\varrho-1)^m \omega^m}\right)^{\varrho-1}.$$

It follows that

$$\varrho\omega \leqq 2(\varrho-1)\omega < \frac{2}{m}\left(m^{\Omega} e^{10\Omega}\right)^{\frac{1}{m}} q^{\frac{1}{m(\varrho-1)}},$$

and that therefore

$$\varrho\omega m^{\Omega+1} e^{6\Omega+5} < \frac{2}{m}\left(m^{\Omega} e^{10\Omega}\right)^{\frac{1}{m}} m^{\Omega+1} e^{6\Omega+5} q^{\frac{1}{m(\varrho-1)}} <$$

$$< \left(m^{\Omega} e^{10\Omega}\right)^{\frac{1}{m}} m^{\Omega} e^{6\Omega+6} q^{\frac{1}{m(\varrho-1)}}.$$

The lemma implies then in this case that

$$\max_{k=1,2,\ldots,m}\left|e^{\frac{\omega_k}{\omega}} - \frac{q_k}{q}\right| > e^{(4\Omega-6)\varrho}\left(m^{\Omega} e^{10\Omega}\right)^{-\frac{m+1}{m}\varrho} q^{-1-\frac{\varrho}{m(\varrho-1)}}.$$

Here we once more use that $\varrho \geqq 2$, hence that

$$q^{-1-\frac{\varrho}{m(\varrho-1)}} = q^{-1-\frac{1}{m}-\frac{1}{m(\varrho-1)}} \geqq q^{-1-\frac{1}{m}} \cdot q^{-\frac{2}{m\varrho}},$$

where, by the choice of ϱ,

$$q^{-\frac{2}{m\varrho}} \geqq \left(\frac{m^{\Omega} e^{10\Omega}}{m^m \varrho^m \omega^m}\right)^{\frac{2}{m}}.$$

Evidently $\Omega \geqq m$, and so, by this inequality,

$$q^{-\frac{2}{m\varrho}} \geqq \frac{e^{20}}{\varrho^2 \omega^2}.$$

Assume, in particular, that also $\Omega \geqq 2$. Then

$$4\Omega - 6 \geqq 2, \quad e^{(4\Omega-6)\varrho} q^{-\frac{2}{m\varrho}} \geqq \frac{e^{2\varrho+20}}{\varrho^2 \omega^2} > \frac{e^{20}}{\omega^2} \quad \text{because} \quad e^{\varrho} > \varrho.$$

Thus, in this second case, we arrive at the estimate

$$\max_{k=1,2,\ldots,m}\left|e^{\frac{\omega_k}{\omega}} - \frac{q_k}{q}\right| > \frac{e^{20}}{\omega^2}\left(m^{\Omega} e^{10\Omega}\right)^{-\frac{m+1}{m}\varrho} q^{-1-\frac{1}{m}}.$$

Our result may be expressed as follows.

Theorem 1. *Let* $\omega, \omega_1, \ldots, \omega_m, q, q_1, \ldots, q_m$, *and* Ω *be* $2m+3$ *integers satisfying the conditions*

$$\omega \geqq 1, \quad q \geqq 1, \quad 0 < \omega_1 < \omega_2 < \cdots < \omega_m = \Omega, \quad \Omega \geqq 2.$$

If

$$\omega \geqq \left(m^{\Omega-m} e^{10\Omega} q\right)^{\frac{1}{m}},$$

then

$$\max_{k=1,2,\ldots,m}\left|e^{\frac{\omega_k}{\omega}} - \frac{q_k}{q}\right| > \left(\omega m^{\Omega+1} e^{6\Omega+5}\right)^{-1} \frac{1}{q}.$$

If, however,

$$\omega < (m^{\Omega - m} e^{10\Omega} q)^{\frac{1}{m}},$$

and if ϱ *denotes the smallest integer satisfying*

$$\left(\frac{m^{\Omega} e^{10\Omega}}{m^m \varrho^m \omega^m}\right)^{\varrho} \leqq \frac{1}{q},$$

then

$$\max_{k=1,2,\ldots,m} \left| e^{\frac{\omega_k}{\omega}} - \frac{q_k}{q} \right| > \frac{e^{20}}{\omega^2} (m^{\Omega} e^{10\Omega})^{-\frac{m+1}{m}\varrho} q^{-1-\frac{1}{m}}.$$

The interest of this theorem lies in the fact that $\omega, \omega_1, \ldots, \omega_m, q, q_1, \ldots, q_m$ may all be variable and are subject only to trivial restrictions. The assertion is particularly strong when $\omega, \omega_1, \ldots, \omega_m$ are fixed, while $q, q_1, \ldots, q_m$ are allowed to tend to infinity. For then the parameter ϱ likewise tends to infinity and is given asymptotically by

$$\varrho \sim \frac{\log q}{\log \log q}.$$

Hence a positive constant c depending only on $\omega, \omega_1, \ldots, \omega_m$ exists so that

$$\max_{k=1,2,\ldots,m} \left| e^{\frac{\omega_k}{\omega}} - \frac{q_k}{q} \right| > q^{-1-\frac{1}{m}-\frac{c}{\log\log q}}$$

for large q.

If also $\omega, \omega_1, \ldots, \omega_m$ are variable, the theorem is much less strong. However, some consequences seem still worth of being mentioned.

12. Theorem 1 implies an analogous theorem on the simultaneous approximations of logarithms. Its proof is based on the following elementary lemma.

Lemma 2. *If* x *and* $y > 0$ *are real numbers such that*

$$|x - \log y| \leqq 1,$$

then

$$|x - \log y| \geqq e^{-x-2} |e^x - y|.$$

Proof. By the mean value theorem,

$$\frac{e^t - 1}{t} = e^{\vartheta t} \quad \text{where} \quad 0 < \vartheta < 1.$$

Hence, on putting $t = x - \log y$,

$$0 < \frac{e^x - y}{x - \log y} = y e^{\vartheta(x - \log y)} \leqq e y.$$

Here

$$\log y \leqq x + 1, \quad y \leqq e^{x+1},$$

whence the assertion.

This lemma we apply to each of the m pairs of numbers

$$x = \frac{\omega_k}{\omega}, \quad y = \frac{q_k}{q} \qquad (k = 1, 2, \ldots, m),$$

for which, evidently,

$$x \leqq \frac{\Omega}{\omega} \leqq \Omega, \quad e^{-x-2} \geqq e^{-\Omega-2}.$$

We next note that Theorem 1 remains valid if the conditions

$$0 < \omega_1 < \omega_2 < \cdots < \omega_m = \Omega$$

are replaced by the weaker hypothesis that the integers $\omega_1, \ldots, \omega_m$ are all distinct and have the maximum Ω. By combining the theorem with the lemma we obtain therefore the following result.

Theorem 2. *Let* $\omega, \omega_1, \ldots, \omega_m, q, q_1, \ldots, q_m, \Omega$ *be* $2m+3$ *positive integers satisfying the conditions*

$$\omega_k \neq \omega_l \quad \text{for} \quad k \neq l; \quad \Omega = \max_{k=1,2,\ldots,m} \omega_k \geqq 2.$$

If ω *satisfies the inequality*

$$\omega \geqq (m^{\Omega-m} e^{10\Omega} q)^{\frac{1}{m}},$$

then

$$\max_{k=1,2,\ldots,m} \left| \log \frac{q_k}{q} - \frac{\omega_k}{\omega} \right| > (\omega m^{\Omega+1} e^{7\Omega+7} q)^{-1}.$$

If, however,

$$\omega < (m^{\Omega-m} e^{10\Omega} q)^{\frac{1}{m}},$$

and if ϱ *denotes the smallest integer satisfying*

$$\left(\frac{m^{\Omega} e^{10\Omega}}{m^m \varrho^m \omega^m} \right)^{\varrho} \leqq \frac{1}{q},$$

then

$$\max_{k=1,2,\ldots,m} \left| \log \frac{q_k}{q} - \frac{\omega_k}{\omega} \right| > \frac{e^{18-\Omega}}{\omega^2} (m^{\Omega} e^{10\Omega})^{-\frac{m+1}{m}\varrho} q^{-1-\frac{1}{m}}.$$

13. We deal in detail with one special application of Theorem 2. For this purpose denote by

$$p_1 = 2, p_2 = 3, \ldots, p_m$$

the first m primes in their natural order. We apply the theorem with

$$q = 1, q_1 = p_1, \ldots, q_m = p_m$$

and choose for $\omega, \omega_1, \ldots, \omega_m$ any $m+1$ positive integers for which the fractions

$\frac{\omega_1}{\omega}, \ldots, \frac{\omega_m}{\omega}$ are approximations of $\log p_1, \ldots, \log p_m$, respectively, that are already so close that

$$\text{(A)} \qquad \max_{k=1,2,\ldots,m} \left| \log p_k - \frac{\omega_k}{\omega} \right| < \frac{1}{2} \log \frac{p_m}{p_{m-1}}.$$

Further put again

$$\Omega = \max(\omega_1, \ldots, \omega_m)$$

and assume that

$$m \geqq 10.$$

From the hypothesis (A),

$$|\log p_k - \log p_l| \geqq \log \frac{p_m}{p_{m-1}} \quad \text{for} \quad k \neq l,$$

and

$$\log p_k \geqq \log 2 > \log \frac{p_m}{p_{m-1}} \quad \text{for all } k.$$

Hence

$$\frac{\omega_{k+1}}{\omega} - \frac{\omega_k}{\omega} = \left(\frac{\omega_{k+1}}{\omega} - \log p_{k+1} \right) + (\log p_{k+1} - \log p_k) + \left(\log p_k - \frac{\omega_k}{\omega} \right)$$

$$> -\frac{1}{2} \log \frac{p_m}{p_{m-1}} + \log \frac{p_m}{p_{m-1}} - \frac{1}{2} \log \frac{p_m}{p_{m-1}} = 0$$

and

$$\frac{\omega_1}{\omega} > \log 2 - \frac{1}{2} \log \frac{p_m}{p_{m-1}} \geqq \log 2 - \frac{1}{2} \log \frac{3}{2} > 0.$$

The hypothesis (A) implies therefore that

$$0 < \omega_1 < \omega_2 < \cdots < \omega_m = \Omega.$$

It also implies that

$$\omega \geqq 2,$$

because, if ω were equal to 1, it would follow that

$$\left| \log p_1 - \frac{\omega_1}{\omega} \right| \geqq |\log 2 - 1| > \frac{1}{2} \log \frac{3}{2} \geqq \frac{1}{2} \log \frac{p_m}{p_{m-1}},$$

for all choices of the integer ω_1, contrary to (A).

Next we have $\omega_m \geqq m \geqq 10$ and therefore

$$\Omega > 2.$$

Thus all conditions of Theorem 2 are satisfied, and this theorem may be applied.

From (A),

$$\Omega < \omega\left(\log p_m + \frac{1}{2}\log\frac{p_m}{p_{m-1}}\right) = \frac{\omega}{2}\log\left(\frac{p_m^3}{p_{m-1}}\right).$$

Here, by Bertrand's law on prime numbers,

$$p_{m-1} > \frac{1}{2}p_m,$$

and by the paper (ROSSER and SCHOENFELD, 1962),

$$p_m < \sqrt{2}\, m \log m.$$

Therefore the quantity Ω allows the upper estimate

$$\Omega < \omega \log(2m \log m).$$

It follows that

$$(m^{\Omega-m} e^{10\Omega})^{\frac{1}{m}} < \frac{1}{m}\exp\left\{\frac{1}{m}(10+\log m)\cdot\omega\log(2m\log m)\right\}.$$

Here the right-hand side does not exceed 2 if

(B) $$\omega \leqq \frac{m\log(2m)}{(10+\log m)\log(2m\log m)},$$

and so, for such values of ω, the second case $\varrho \geqq 2$ of Theorem 2 cannot hold. Therefore, by this theorem,

$$\max_{k=1,2,\ldots,m}\left|\log p_k - \frac{\omega_k}{\omega}\right| > (\omega m^{\Omega+1} e^{7\Omega+7})^{-1}.$$

In this estimate,

$$\begin{aligned}\omega m^{\Omega+1} e^{7\Omega+7} &< e^7 m\omega\exp\{(7+\log m)\cdot\omega\log(2m\log m)\}\\ &\leqq e^7 m\frac{m\log(2m)}{(10+\log m)\log(2m\log m)}\exp\left\{\frac{m(7+\log m)\log(2m)}{10+\log m}\right\}\\ &< e^7\frac{m^2}{\log m}\exp\{m\log(2m)\},\end{aligned}$$

where, by $m \geqq 10$,

$$e^7\frac{m^2}{\log m} < 2{,}000\frac{m^2}{2} < (2m)^5.$$

Hence it follows from (B) that

$$\max_{k=1,2,\ldots,m}\left|\log p_k - \frac{\omega_k}{\omega}\right| > (2m)^{-m-5}.$$

A stronger result is obtained if ω is restricted to the smaller range

(C) $$\omega \leqq \frac{m}{(7+\log m)\log(2m\log m)}.$$

Now

$$\omega m^{\Omega+1} e^{7\Omega+7} < e^7 m \cdot \frac{m}{(7+\log m)\log(2m\log m)} \cdot e^m < \frac{e^7 m^2 e^m}{(\log m)^2},$$

where, by $m \geqq 10$,

$$\frac{e^7 m^2 e^m}{(\log m)^2} < \frac{2000\, m^2 e^m}{2^2} < m^5 e^m .$$

It follows thus from (C) that

$$\max_{k=1,2,\ldots,m} \left| \log p_m - \frac{\omega_m}{\omega} \right| > m^{-5} e^{-m} .$$

The two right-hand sides

$$(2m)^{-m-5} \quad \text{and} \quad m^{-5} e^{-m}$$

in the estimates just established are smaller than the right-hand side

$$\frac{1}{2} \log \frac{p_m}{p_{m-1}}, \quad = \lambda \text{ say},$$

of the hypothesis (A). For

$$p_m < \sqrt{2}\, m \log m < m^2 ,$$

because

$$\log m \leqq \log 2 + \frac{1}{2}(m-2) < \frac{m}{2} .$$

Therefore

$$\lambda \geqq \frac{1}{2} \log \frac{p_m}{p_m - 1} > \frac{1}{2} \log \frac{m^2}{m^2 - 1} > \frac{1}{2} \log(1 + m^{-2}),$$

where, by $m \geqq 10$,

$$\frac{1}{2} \log(1 + m^{-2}) > \frac{1}{2}(m^{-2} - m^{-4} - m^{-6} - \cdots) > \frac{1}{3} m^{-2} .$$

Hence

$$\lambda > \frac{1}{6} m^{-2} > m^{-3},$$

giving the assertion easily.

We may then omit again the hypothesis (A), and we are also allowed in including the trivial denominator $\omega = 1$. Then, on combining the preceeding results, we obtain the following theorem.

Theorem 3. *Let $m \geqq 10$; let $p_1 = 2$, $p_2 = 3, \ldots, p_m$ be the first m primes; and let $\omega, \omega_1, \ldots, \omega_m$ be $m+1$ positive integers. Then*

$$\max_{k=1,2,\ldots,m} \left| \log p_k - \frac{\omega_k}{\omega} \right| > (2m)^{-m-5} \quad \textit{if} \quad 1 \leqq \omega \leqq \frac{m \log(2m)}{(10 + \log m)\log(2m \log m)},$$

and

$$\max_{k=1,2,\ldots,m}\left|\log p_k-\frac{\omega_k}{\omega}\right|>m^{-5}e^{-m}\quad \text{if}\quad 1\leqq\omega\leqq\frac{m}{(7+\log m)\log(2m\log m)}.$$

These two inequalities are rather weak, but it does not seem to be easy to obtain much better ones. For larger values of ω the position is worse.

14. Next put

$$\omega_1=1,\omega_2=2,\ldots,\omega_m=m,\quad \text{hence}\quad \Omega=m.$$

The general estimates for $a_{hk}^{(j)}$, $a_{hk}(z)$, and $r_h(z)$ can in this special case be a little improved. For now evidently

$$M_k=k!\,(m-k)!=m!\binom{m}{k}^{-1}\geqq 2^{-m}m!,\quad M=m!,$$

and by the paper (Rosser and Schoenfeld, 1962),

$$N\leqq e^{1.04m}.$$

The formulae in § 4 become therefore

$$|a_{hk}^{(j)}|\leqq 2^{m\varrho}\,e^{1.04m(\varrho+1)}\,\varrho!\,m^{-m\varrho}(m+1)^{(m+1)\varrho},$$

$$|a_{hk}(z)|\leqq 2^{m\varrho}\,e^{1.04m(\varrho+1)}\,\varrho!\,m^{-m\varrho}(m+1)^{(m+1)\varrho}\,e^{|z|},$$

$$|r_h(z)|\leqq (m!)^{\varrho-1}\,e^{1.04m(\varrho+1)}(m+1)^{-(m+1)(\varrho-1)}\{(\varrho-1)!\}^{-m}|z|^{(m+1)\varrho}\,e^{m|z|}.$$

These estimates can be further simplified if we assume from now on that m is already sufficiently large, but that ϱ may be any positive integer, small or large. For

$$\varrho!\leqq e\sqrt{\varrho}\,\varrho^{\varrho}\,e^{-\varrho},\quad \varrho^{\frac{1}{\varrho}}\leqq 3^{\frac{1}{3}},\quad \left(1+\frac{1}{m}\right)^m\leqq e,$$

while

$$(m+1)^{\frac{1}{m}}>1$$

becomes arbitrarily close to 1. Since $2e^{1.04}<e^{1.74}$, it follows that

$$|a_{hk}^{(j)}|\leqq 2^{m\varrho}\,e^{1.04m(\varrho+1)}\cdot e\sqrt{\varrho}\,\varrho^{\varrho}\,e^{-\varrho}(m+1)^{\varrho}\left(1+\frac{1}{m}\right)^{m\varrho}\leqq$$

$$\leqq\{2e^{1.04\frac{\varrho+1}{\varrho}}\,e^{\frac{1}{m\varrho}}\,\varrho^{\frac{1}{2m\varrho}}(m+1)^{\frac{1}{m}}\}^{m\varrho}\,\varrho^{\varrho}<e^{1.75m(\varrho+1)}\,\varrho^{\varrho}$$

and hence

$$(1)\qquad \boxed{|a_{hk}^{(j)}|<e^{1.75m(\varrho+1)}\,\varrho^{\varrho},\quad |a_{hk}(z)|<e^{1.75m(\varrho+1)}\,\varrho^{\varrho}\,e^{|z|}.}$$

15. Next, the estimate for $r_h(z)$ may be written as

$$|r_h(z)|\leqq R\,|z|^{(m+1)\varrho}\,e^{m|z|},$$

where R denotes the expression

$$R=(m!)^{\varrho-1}\,e^{1.04m(\varrho+1)}(m+1)^{-(m+1)(\varrho-1)}\{(\varrho-1)!\}^{-m}$$

which does not depend on z. Since

$$m!\leqq e\sqrt{m}\,m^m\,e^{-m},\quad (\varrho-1)!\geqq\sqrt{\frac{2\pi}{\varrho}}\,\varrho^\varrho\,e^{-\varrho},\quad \left(1+\frac{1}{m}\right)^{m+1}\geqq e,$$

we find that

$$R\leqq e^{\varrho-1}\,m^{\frac{\varrho-1}{2}}\,m^{m(\varrho-1)}\,e^{-m(\varrho-1)}\cdot e^{1.04m(\varrho+1)}(m+1)^{-(m+1)(\varrho-1)}\cdot\left(\frac{\varrho}{2\pi}\right)^{\frac{m}{2}}\varrho^{-m\varrho}\,e^{m\varrho}.$$

Here

$$m^{m(\varrho-1)}(m+1)^{-(m+1)(\varrho-1)}=m^{-(\varrho-1)}\left(1+\frac{1}{m}\right)^{-(m+1)(\varrho-1)}\leqq m^{-(\varrho-1)}\,e^{-(\varrho-1)},$$

so that after a trivial simplification,

$$R\leqq e^{(\varrho-1)-m(\varrho-1)+1.04m(\varrho+1)+m\varrho-(\varrho-1)}\,m^{-\frac{\varrho-1}{2}}\left(\frac{\varrho}{2\pi}\right)^{\frac{m}{2}}\varrho^{-m\varrho}\leqq$$

$$\leqq e^{m+1.04m(\varrho+1)}\,m^{-\frac{\varrho-1}{2}}\left(\frac{\varrho}{2\pi}\right)^{\frac{m}{2}}\varrho^{-m\varrho}.$$

On omitting the factors that are smaller than 1,

$$R<e^{1.04m(\varrho+2)}\,\varrho^{-m(\varrho-\frac{1}{2})},$$

whence

$$|r_h(z)|<e^{1.04m(\varrho+2)}\,\varrho^{-m(\varrho-\frac{1}{2})}|z|^{(m+1)\varrho}\,e^{m|z|}. \tag{2}$$

If also ϱ is sufficiently large, this inequality can be further simplified to

$$|r_h(z)|<e^{1.05m\varrho}\,\varrho^{-m\varrho}\,|z|^{(m+1)\varrho}\,e^{m|z|}. \tag{3}$$

16. As a first application of the last estimates, let g be a very large positive integer, and let γ be the integer defined by

$$e^g=\gamma+\delta,\quad\text{where}\quad -\frac{1}{2}\leqq\delta<+\frac{1}{2}.$$

In the identity

$$r_h(z)=\sum_{k=0}^{m}a_{hk}(z)\,e^{kz}$$

substitute

$$z=g,\quad e^z=\gamma+\delta.$$

Then

$$r_h(g)=\sum_{k=0}^{m}a_{hk}(g)\,(\gamma+\delta)^k=\sum_{k=0}^{m}\sum_{l=0}^{k}a_{hk}(g)\binom{k}{l}\gamma^{k-l}\,\delta^l,$$

or, say,

$$r_h(g) = \sum_{l=0}^{m} b_{hl}\,\delta^l$$

where b_{hl} denotes the expression

$$b_{hl} = \sum_{k=l}^{m} a_{hk}(g)\binom{k}{l}\gamma^{k-l}.$$

In particular,

$$b_{h0} = \sum_{k=0}^{m} a_{hk}(g)\,\gamma^k.$$

Here, by § 4, the determinant $d(g)$ with the elements $a_{hk}(g)$ does not vanish. Hence a suffix h exists for which

$$b_{h0} \neq 0.$$

Let h from now on be chosen in this manner.

17. Since

$$a_{hk}(z) = \sum_{j=0}^{\varrho} a_{hk}^{(j)}\,z^j,$$

we have

$$b_{hl} = \sum_{k=l}^{m}\sum_{j=0}^{\varrho} a_{hk}^{(j)}\,g^j\binom{k}{l}\gamma^{k-l},$$

so that b_{hl} is an integer. By the estimate (1),

$$|a_{hk}^{(j)}| < e^{1.75m(\varrho+1)}\,\varrho^\varrho.$$

Further

$$\sum_{j=0}^{\varrho} g^j \leqq (g+1)^\varrho, \quad \binom{k}{l} \leqq 2^k, \quad \sum_{k=l}^{m}\binom{k}{l} \leqq \sum_{k=0}^{m} 2^k < 2^{m+1}.$$

Hence, for all suffices l,

$$|b_{hl}| < e^{1.75m(\varrho+1)}\,\varrho^\varrho(g+1)^\varrho\,2^{m+1}\,\gamma^m.$$

On the other hand, b_{h0} is a non-vanishing integer, and hence

$$|b_{h0}| \geqq 1.$$

Let us assume for the moment that

$$|\delta| < \frac{1}{3}\{e^{1.75m(\varrho+1)}\,\varrho^\varrho(g+1)^\varrho\,2^{m+1}\,\gamma^m\}^{-1}$$

and hence that

$$|\delta| < \frac{1}{3}.$$

We find then that

$$|r_h(g)| \geqq |b_{h0}| - |\delta| \sum_{l=1}^{m} |b_{hl}| \, |\delta|^{l-1} >$$

$$> 1 - \frac{1}{3}\{e^{1.75m(\varrho+1)} \varrho^{\varrho}(g+1)^{\varrho} 2^{m+1} \gamma^m\}^{-1} \cdot \sum_{l=1}^{m} e^{1.75m(\varrho+1)} \varrho^{\varrho}(g+1)^{\varrho} 2^{m+1} \gamma^m \cdot \left(\frac{1}{3}\right)^{l-1}.$$

Here

$$\frac{1}{3} \sum_{l=1}^{\infty} \left(\frac{1}{3}\right)^{l-1} = \frac{1}{2},$$

and so it follows that

$$|r_h(g)| > \frac{1}{2}.$$

However, if both m and ϱ are sufficiently large, then, by (3),

$$|r_h(g)| < e^{1.05m\varrho} \, \varrho^{-m\varrho} \, g^{(m+1)\varrho} \, e^{mg}.$$

If now m and ϱ are chosen so as to satisfy the inequality

$$\text{(D)} \qquad e^{1.05m\varrho} \, \varrho^{-m\varrho} \, g^{(m+1)\varrho} \, e^{mg} \leqq \frac{1}{2},$$

a contradiction arises. The assumed upper bound for δ was therefore false, and so (D) implies instead the lower bound

$$\text{(E)} \qquad |\delta| \geqq \frac{1}{3} \{e^{1.75m(\varrho+1)} \, \varrho^{\varrho}(g+1)^{\varrho} \, 2^{m+1} \, \gamma^m\}^{-1}.$$

Denote by α and β two positive absolute constants to be selected immediately, and take for m and ϱ the integers

$$m = [\alpha \log g] + 1, \quad \varrho = [\beta g] + 1,$$

where, as usual, $[x]$ is the integral part of x. Then m and ϱ will exceed any given bounds as soon as g is sufficiently large, and so, under this hypothesis, we were justified in applying the formula (3).

The inequality (D) is equivalent to

$$\varrho \geqq e^{1.05} \, g^{1+\frac{1}{m}} \, e^{\frac{g}{\varrho}} \, 2^{\frac{1}{m\varrho}}.$$

Here, by our choice of m and ϱ,

$$m > \alpha \log g, \quad \varrho > \beta g,$$

and therefore

$$g^{\frac{1}{m}} < e^{\frac{1}{\alpha}}, \quad e^{\frac{g}{\varrho}} < e^{\frac{1}{\beta}}.$$

The remaining factor

$$2^{\frac{1}{m\varrho}}$$

is arbitrarily close to 1 as soon as g is sufficiently large. Thus, for such g,

$$e^{1.05} \, g^{1+\frac{1}{m}} \, e^{\frac{g}{\varrho}} \, 2^{\frac{1}{m\varrho}} < e^{1.06+\frac{1}{\alpha}+\frac{1}{\beta}} \, g.$$

Assume now that

(F) $$\beta \geqq e^{1.06+\frac{1}{\alpha}+\frac{1}{\beta}}.$$

The condition (D) is then satisfied because

$$\varrho > \beta g \geqq e^{1.06+\frac{1}{\alpha}+\frac{1}{\beta}} g > e^{1.05} g^{1+\frac{1}{m}} e^{\frac{g}{\varrho}} 2^{\frac{1}{m\varrho}}.$$

Also, for all sufficiently large g,

$$e^{1.75m(\varrho+1)} < e^{1.76\alpha\beta g \log g}, \quad \varrho^{\varrho} < (\beta g)^{1.005\beta g} < e^{1.01\beta g \log g},$$

$$(g+1)^{\varrho} < e^{1.01\beta g \log g}, \quad \gamma^m < \left(e^g + \frac{1}{2}\right)^{\alpha \log g + 1} < e^{1.01\alpha g \log g},$$

$$3 \times 2^{m+1} < e^{\alpha \log g} < e^{0.01\alpha g \log g}.$$

The lower bound (E) for δ takes therefore the form

$$|\delta| > e^{-(1.76\alpha\beta g \log g + 1.01\beta g \log g + 1.01\beta g \log g + 0.01\alpha g \log g + 1.01\alpha g \log g)},$$

that is,

$$|\delta| > g^{-(1.76\alpha\beta + 2.02\beta + 1.02\alpha)g}.$$

We finally fix the constants α and β so that (F) is satisfied, while at the same time the sum

$$\sigma = 1.76\alpha\beta + 2.02\beta + 1.02\alpha$$

becomes small. After some numerical work one is led to the values

$$\beta = 7, \quad \alpha = 1.35,$$

when

$$e^{1.06+\frac{1}{\alpha}+\frac{1}{\beta}} < e^{1.945} < 7, \quad \sigma < 32.4.$$

We arrive thus at the following result.

Theorem 4. *Let g be any sufficiently large positive integer, and let γ be the integer closest to e^g. Then*

$$|e^g - \gamma| > g^{-33g}.$$

By means of more careful estimates, the constant 33 in this theorem can be a little decreased. However, it does not seem to be easy to obtain any essential improvement of the theorem. Previously, by means of a different method, I had proved the analogous estimate with 40 instead of 33 for the constant (Mahler, 1953).

18. We finally apply the formulae (1) and (2) to the study of the rational approximations of π.

Denote by p and q any two positive integers such that

$$\pi = \frac{p}{q} + \delta, \quad \text{where} \quad -\frac{1}{2q} \leqq \delta < +\frac{1}{2q}.$$

It is trivial that there exists arbitrarily large integers of this kind.

In the identity

$$r_h(z) = \sum_{k=0}^{m} a_{hk}(z) e^{kz}$$

put now

$$z = \frac{\pi i}{2}, \quad e^z = i,$$

so that

$$r_h\left(\frac{\pi i}{2}\right) = \sum_{k=0}^{m} a_{hk}\left(\frac{\pi i}{2}\right) i^k.$$

Here

$$a_{hk}\left(\frac{\pi i}{2}\right) = \sum_{j=0}^{\varrho} a_{hk}^{(j)} \left(\frac{i}{2}\right)^j \left(\frac{p}{q} + \delta\right)^j = \sum_{j=0}^{\varrho} \sum_{l=0}^{j} \binom{j}{l} a_{hk}^{(j)} \left(\frac{i}{2}\right)^j \left(\frac{p}{q}\right)^{j-l} \delta^l,$$

or, say,

$$a_{hk}\left(\frac{\pi i}{2}\right) = \sum_{l=0}^{\varrho} c_{hkl}\, \delta^l$$

where

$$c_{hkl} = \sum_{j=l}^{\varrho} \binom{j}{l} a_{hk}^{(j)} \left(\frac{i}{2}\right)^j \left(\frac{p}{q}\right)^{j-l}.$$

Therefore

$$r_h\left(\frac{\pi i}{2}\right) = \sum_{k=0}^{m} \sum_{l=0}^{\varrho} c_{hkl}\, i^k \delta^l = \sum_{l=0}^{\varrho} C_{hl}\, \delta^l$$

where we have put

$$C_{hl} = \sum_{k=0}^{m} c_{hkl}\, i^k.$$

In particular,

$$C_{h0} = \sum_{k=0}^{m} c_{hk0}\, i^k = \sum_{k=0}^{m} \sum_{j=0}^{\varrho} a_{hk}^{(j)} \left(\frac{i}{2}\right)^j \left(\frac{p}{q}\right)^j i^k = \sum_{k=0}^{m} a_{hk}\left(\frac{ip}{2q}\right) i^k.$$

Here, similarly as before, the determinant $d\left(\frac{ip}{2q}\right)$ with the elements $a_{hk}\left(\frac{ip}{2q}\right)$ does not vanish. Therefore, from now on, we may again assume that h is chosen so as to satisfy the inequality

$$C_{h0} \neq 0.$$

19. The expressions

$$(2q)^\varrho\, c_{hkl} = \sum_{j=l}^{\varrho} \binom{j}{l} a_{hk}^{(j)}\, 2^{\varrho - j} q^{\varrho-(j-l)} i^j p^{j-l}$$

and

$$(2q)^\varrho C_{hl} = \sum_{k=0}^{m} (2q)^\varrho c_{hkl} i^k$$

are integers in the Gaussian field $Q(i)$. In particular, $(2q)^\varrho C_{h0}$ is a Gaussian integer different from zero, and hence its absolute value is not less than 1. Therefore

$$|C_{h0}| \geqq (2q)^{-\varrho}.$$

Assume now that m is already very large, while no such restriction need be imposed on ϱ. We are thus allowed to make use of the estimates (1) and (2). Since

$$\binom{j}{l} \leqq 2^j, \quad \binom{\varrho - l}{j - l} \geqq 1 \quad \text{for} \quad l \leqq j \leqq \varrho,$$

it follows from (1) that

$$|c_{hkl}| < \sum_{j=l}^{\varrho} \binom{\varrho - l}{j - l} 2^j e^{1.75m(\varrho+1)} \varrho^\varrho \left(\frac{1}{2}\right)^j \left(\frac{p}{q}\right)^{j-l}.$$

Here

$$\frac{p}{q} \leqq \pi + \delta \leqq \pi + \frac{1}{2} < 4,$$

and therefore

$$|c_{hkl}| < e^{1.75m(\varrho+1)} \varrho^\varrho \cdot 5^{\varrho - l}$$

and

$$|C_{hl}| < \frac{m+1}{5^l} e^{1.75m(\varrho+1)} \varrho^\varrho \cdot 5^\varrho.$$

Let now, for the moment, δ be so small that

$$|\delta| < \{(m+1) e^{1.75m(\varrho+1)} \varrho^\varrho 5^\varrho (2q)^\varrho\}^{-1},$$

hence that

$$|\delta| < 1.$$

Then

$$\left| r_h\left(\frac{\pi i}{2}\right) \right| \geqq |C_{h0}| - |\delta| \sum_{l=1}^{m} |C_{hl}| \, |\delta|^{l-1} >$$

$$> (2q)^{-\varrho} - \{(m+1) e^{1.75m(\varrho+1)} \varrho^\varrho (10q)^\varrho\}^{-1} \sum_{l=1}^{m} \frac{m+1}{5^l} e^{1.75m(\varrho+1)} \varrho^\varrho 5^\varrho,$$

and since

$$\sum_{l=1}^{\infty} \left(\frac{1}{5}\right)^l = \frac{1}{4},$$

it follows that

$$\left| r_h\left(\frac{\pi i}{2}\right) \right| > \frac{3}{4} (2q)^{-\varrho}.$$

On the other hand, by (2),

$$\left|r_h\left(\frac{\pi i}{2}\right)\right| < e^{1.04m(\varrho+2)}\,\varrho^{-m(\varrho-\frac{1}{2})}\left(\frac{\pi}{2}\right)^{(m+1)\varrho} e^{\frac{m\pi}{2}}.$$

Hence, if m and ϱ satisfy the inequality

$$\text{(G)} \qquad e^{1.04m(\varrho+2)}\,\varrho^{-m(\varrho-\frac{1}{2})}\left(\frac{\pi}{2}\right)^{(m+1)\varrho} e^{\frac{m\pi}{2}} \leqq \frac{3}{4}(2q)^{-\varrho},$$

a contradiction arises, showing that the assumed lower bound for δ cannot be valid. The inequality (G) implies therefore that, on the contrary,

$$\text{(H)} \qquad |\delta| \geqq \{(m+1)\,e^{1.75m(\varrho+1)}\,\varrho^{\varrho}(10q)^{\varrho}\}^{-1}.$$

20. From now on let q be very large. If m is then defined by

$$m = [\lambda \log q] + 1$$

where $\lambda > 0$ is a constant to be selected immediately, also m will be arbitrarily large, as required in the preceeding proof.

The inequality (G) is equivalent to

$$\varrho \geqq \left(\frac{4}{3}\right)^{\frac{1}{m(\varrho-\frac{1}{2})}} e^{1.04\frac{\varrho+2}{\varrho-\frac{1}{2}}}\left(\frac{\pi}{2}\right)^{\frac{(m+1)\varrho}{m(\varrho-\frac{1}{2})}} e^{\frac{\pi}{2\varrho-1}}\,(2q)^{\frac{\varrho}{m(\varrho-\frac{1}{2})}}.$$

Here the factors

$$\left(\frac{4}{3}\right)^{\frac{1}{m(\varrho-\frac{1}{2})}} \quad \text{and} \quad 2^{\frac{\varrho}{m(\varrho-\frac{1}{2})}}$$

are arbitrarily close to 1 and so may be assumed to have a product

$$\left(\frac{4}{3}\right)^{\frac{1}{m(\varrho-\frac{1}{2})}} 2^{\frac{\varrho}{m(\varrho-\frac{1}{2})}} < e^{0.01\frac{\varrho+2}{\varrho-\frac{1}{2}}}.$$

We may similarly demand that

$$\left(\frac{\pi}{2}\right)^{\frac{(m+1)\varrho}{m(\varrho-\frac{1}{2})}} < \left(\frac{\pi}{2}\right)^{1.01\frac{\varrho}{\varrho-\frac{1}{2}}}.$$

Next

$$\frac{1}{m} < \frac{1}{\lambda \log q},$$

and hence

$$q^{\frac{\varrho}{m(\varrho-\frac{1}{2})}} < e^{\frac{\varrho}{\lambda(\varrho-\frac{1}{2})}}.$$

Thus (G) is for large q certainly satisfied if

$$\varrho \geqq e^{1.05\frac{\varrho+2}{\varrho-\frac{1}{2}}}\left(\frac{\pi}{2}\right)^{1.01\frac{\varrho}{\varrho-\frac{1}{2}}} e^{\frac{\pi}{2\varrho-1}}\, e^{\frac{\varrho}{\lambda(\varrho-\frac{1}{2})}},$$

that is, if

$$\varrho \geqq e^{\left(1.05\varrho + 2.1 + 1.01\varrho \log\frac{\pi}{2} + \frac{\pi}{2} + \frac{\varrho}{\lambda}\right)\frac{1}{\varrho - \frac{1}{2}}}.$$

Here

$$\frac{\pi}{2} < 1.571, \quad \log\frac{\pi}{2} < 0.452, \quad 1.01\log\frac{\pi}{2} < 0.457.$$

The condition for ϱ is therefore satisfied if

$$\varrho \geqq e^{\left(3,671 + 1,507\varrho + \frac{\varrho}{\lambda}\right)\frac{1}{\varrho - \frac{1}{2}}},$$

or equivalent to this, if

$$\frac{\varrho}{\lambda} \leqq \left(\varrho - \frac{1}{2}\right)\log\varrho - (3.671 + 1.507\varrho).$$

This inequality again is easily seen to hold if

$$\varrho = 14, \quad \lambda = 1.35.$$

On substituting the values

$$\varrho = 14, \quad m = [1.35\log q] + 1$$

in (H), we finally obtain for large q the following result.

Theorem 5. *If p and q are positive integers and q is sufficiently large, then*

$$\left|\pi - \frac{p}{q}\right| > q^{-45}.$$

This result is not as strong as the estimate

$$\left|\pi - \frac{p}{q}\right| > q^{-42} \quad \text{for} \quad q \geqq 2$$

which I have previously obtained by means of a different method (MAHLER, 1953a).

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