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ON A CLASS OF ENTIRE FUNCTIONS

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In his well-known book „Transzendentnye i algebraitcheskie tchisla", Moskva 1952, pp. 175—181, A. O. GELFOND investigated in detail properties of functions

$$E(z)=\sum_{\nu=0}^{n-1}\sum_{\mu=0}^{m_\nu-1}A_{\mu\nu}z^\mu e^{\alpha_\nu z}$$

where the α_ν are distinct complex numbers, and the coefficients $A_{\mu\nu}$ are arbitrary complex constants. In the present paper I continue his investigations a little further and prove a general theorem which may have some interest in itself. In order to make the paper self-contained, I have repeated some of Gelfond's proofs.

1. Let $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ be finitely many distinct complex numbers; let $m_0, m_1, \dots, m_{n-1}$ be an equal number of positive integers; and let

$$Q(\zeta)=\prod_{\nu=0}^{n-1}(\zeta-\alpha_\nu)^{m_\nu}, \quad m=\sum_{\nu=0}^{n-1}m_\nu.$$

Let further M and N be two integers satisfying

$$0\leqq M\leqq m_N-1, \quad 0\leqq N\leqq n-1.$$

Denote by I the integral

$$I=\frac{1}{2\pi i}\int_C\frac{(\zeta-\alpha_N)^M\,d\zeta}{(\zeta-z)Q(\zeta)}$$

where C is a circle in the ζ-plane with centre $\zeta=0$ and of so large a radius that it contains the $n+1$ points $\alpha_0, \alpha_1, \dots, \alpha_{n-1}, z$ in its interior. The integrand of I is a rational function of ζ which has at $\zeta=\infty$ a zero of order

$$m+1-M\geqq m+1-(m_N-1)\geqq 2.$$

The residue at $\zeta=\infty$ is therefore equal to zero, and hence

(1) $$I\equiv 0 \quad \text{identically in } z.$$

2. Assume for the moment that z is distinct from $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$, and denote by $C_0, C_1, \dots, C_{n-1}, C_n$ circles of very small radii with centres at $\alpha_0, \alpha_1, \dots, \alpha_{n-1}, z$, respectively. By Cauchy's theorem,

(2) $$I = \sum_{\nu=0}^{n} \frac{1}{2\pi i} \int_{C_\nu} \frac{(\zeta-\alpha_N)^M \, d\zeta}{(\zeta-z)Q(\zeta)}, \quad = \sum_{\nu=0}^{n} I_\nu \quad \text{say.}$$

The integrals I_ν may be written in the form

$$I_\nu = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{C_\nu} \frac{(\zeta-\alpha_\nu)^{m_\nu}(\zeta-\alpha_N)^M}{(\zeta-z)Q(\zeta)} \cdot \frac{d\zeta}{(\zeta-\alpha_\nu)^{m_\nu}} & \text{if } \; 0 \leqq \nu \leqq m-1, \; \nu \neq N, \\ \dfrac{1}{2\pi i} \displaystyle\int_{C_N} \frac{(\zeta-\alpha_N)^{m_N}}{(\zeta-z)Q(\zeta)} \cdot \frac{d\zeta}{(\zeta-\alpha_N)^{m_N-M}} & \text{if } \; \nu = N, \\ \dfrac{1}{2\pi i} \displaystyle\int_{C_n} \frac{(\zeta-\alpha_N)^M}{Q(\zeta)} \cdot \frac{d\zeta}{\zeta-z} & \text{if } \; \nu = n, \end{cases}$$

where the first factors of the integrands are regular at the points $\zeta=\alpha_\nu$, $\zeta=\alpha_N$, and $\zeta=z$, respectively. Hence, by the residue theorem, these integrals have the explicit values,

$$I_\nu = \begin{cases} \dfrac{1}{(m_\nu-1)!} \left(\dfrac{d}{d\zeta}\right)^{m_\nu-1} \left\{\dfrac{(\zeta-\alpha_\nu)^{m_\nu}(\zeta-\alpha_N)^M}{(\zeta-z)Q(\zeta)}\right\}_{\zeta=\alpha_\nu} & \text{if } \; 0 \leqq \nu \leqq n-1, \; \nu \neq N, \\ \dfrac{1}{(m_N-M-1)!} \left(\dfrac{d}{d\zeta}\right)^{m_N-M-1} \left\{\dfrac{(\zeta-\alpha_N)^{m_N}}{(\zeta-z)Q(\zeta)}\right\}_{\zeta=\alpha_N} & \text{if } \; \nu = N, \\ \left\{\dfrac{(\zeta-\alpha_N)^M}{Q(\zeta)}\right\}_{\zeta=z} & \text{if } \; \nu = n. \end{cases}$$

Therefore, on putting

$$P_{MN}(z) = -\frac{Q(z)I_N}{M!} = \frac{Q(z)}{M!(m_N-M-1)!} \left(\frac{d}{d\zeta}\right)^{m_N-M-1} \left\{\frac{(\zeta-\alpha_N)^{m_N}}{(z-\zeta)Q(\zeta)}\right\}_{\zeta=\alpha_N}$$

and

$$p_{MN}(z) = \frac{Q(z)}{M!} \sum_{\substack{\nu=0 \\ \nu\neq N}}^{n-1} I_\nu = \frac{Q(z)}{M!} \sum_{\substack{\nu=0 \\ \nu\neq N}}^{n-1} \frac{1}{(m_\nu-1)!} \left(\frac{d}{d\zeta}\right)^{m_\nu-1} \left\{\frac{(\zeta-\alpha_\nu)^{m_\nu}(\zeta-\alpha_N)^M}{(\zeta-z)Q(\zeta)}\right\}_{\zeta=\alpha_\nu},$$

it follows from (1) and (2) that

(3) $$P_{MN}(z) = \frac{(z-\alpha_N)^M}{M!} + p_{MN}(z) \quad \text{identically in } z.$$

From the explicit expressions it is easily seen that $P_{MN}(z)$ and $p_{MN}(z)$ are polynomials in z at most of degree

$$m-1=\sum_{\nu=0}^{n-1} m_\nu - 1,$$

and that

$$P_{MN}(z) \quad \text{is divisible by} \quad \prod_{\substack{\nu=0\\ \nu\neq N}}^{n-1} (z-\alpha_\nu)^{m_\nu},$$

$$p_{MN}(z) \quad \text{is divisible by} \quad (z-\alpha_N)^{m_N}.$$

Hence, from (3), it follows that

$$\text{(4)} \qquad P_{MN}^{(\mu)}(\alpha_\nu) = \begin{cases} 1 & \text{if} \quad \mu=M,\ \nu=N, \\ 0 & \text{if} \quad (\mu-M)^2+(\nu-N)^2>0,\ 0\leqq \mu \leqq m_\nu-1,\ 0\leqq \nu\leqq n-1. \end{cases}$$

In the trivial case $n=1$ we also see that $N=0$ and

$$P_{M0}(z)=\frac{(z-\alpha_0)^M}{M!} \quad \text{if} \quad 0\leqq M\leqq m_0-1.$$

3. Since the degree of $P_{MN}(z)$ does not exceed $m-1$, this polynomial can be written as

$$P_{MN}(z)=\sum_{l=0}^{m-1} P_{MN}^{(l)} z^l.$$

Our next aim is to obtain upper bounds for these coefficients. This will be done by first estimating upper bounds for $|P_{MN}(z)|$.

Put, for shortness,

$$a=\max_{0\leqq\nu\leqq n-1}(|\alpha_\nu|,1), \quad a_1=\min_{0\leqq N\leqq n-1}\prod_{\substack{\nu=0\\ \nu\neq N}}^{n-1}|\alpha_N-\alpha_\nu|^{m_\nu/m},$$

$$a_2=\min_{\substack{0\leqq\nu\leqq n-1\\ 0\leqq N\leqq n-1\\ \nu\neq N}}\left(|\alpha_N-\alpha_\nu|^{m_N/m},1\right);$$

in the trivial case $n=1$ these constants become,

$$a=\max(|\alpha_0|,1), \quad a_1=1, \quad a_2=1.$$

The definitions imply that

$$a\geqq 1\geqq a_2$$

and

$$\prod_{\substack{\nu=0\\ \nu\neq N}}^{n-1}|\alpha_N-\alpha_\nu|^{m_\nu}\geqq a_1^m, \quad |\alpha_N-\alpha_\nu|^{m_N}\geqq a_2^m \quad \text{if} \quad N\neq\nu.$$

From the expression for $P_{MN}(z)$ in terms of I_N,

$$P_{MN}(z)=\frac{Q(z)}{2\pi i M!}\int_{C_N}\frac{(\zeta-\alpha_N)^M\,d\zeta}{(z-\zeta)Q(\zeta)}.$$

We choose for C_N the circle

$$|\zeta-\alpha_N|=\frac{1}{2}a_2^{m/m_N},$$

and assume now that ζ lies on C_N and that

$$|z|=2a.$$

It is obvious that C_N encloses none of the points α_ν where $\nu\neq N$. Nor does it enclose the point z because

$$|\zeta|\leqq|\alpha_N|+\frac{1}{2}a_2^{m/m_N}\leqq a+\frac{1}{2}a=\frac{3}{2}a.$$

It furthermore follows that

$$|z-\zeta|\geqq 2a-\frac{3}{2}a=\frac{a}{2}\geqq\frac{1}{2}.$$

Next, if $\nu\neq N$,

$$|\zeta-\alpha_N|=\frac{1}{2}a_2^{m/m_N}\leqq\frac{1}{2}|\alpha_N-\alpha_\nu|,$$

hence

$$|\zeta-\alpha_\nu|=|(\alpha_N-\alpha_\nu)+(\zeta-\alpha_N)|\geqq\frac{1}{2}|\alpha_N-\alpha_\nu|$$

whence

$$\left|\frac{(\zeta-\alpha_N)^{m_N}}{Q(\zeta)}\right|\leqq\prod_{\substack{\nu=0\\ \nu\neq N}}^{n-1}|\zeta-\alpha_\nu|^{-m_\nu}\leqq\prod_{\substack{\nu=0\\ \nu\neq N}}^{n-1}\left\{\frac{1}{2}|\alpha_N-\alpha_\nu|\right\}^{-m_\nu}\leqq 2^{m-m_N}a_1^{-m}.$$

Further

$$\left|\frac{(\zeta-\alpha_N)^{M-m_N}}{z-\zeta}\right|\leqq 2\cdot\left(\frac{1}{2}a_2^{m/m_N}\right)^{M-m_N}=2^{m_N-M+1}a_2^{-m+(mM/m_N)}.$$

It follows therefore that

$$\left|\frac{1}{2\pi i}\int_{C_N}\frac{(\zeta-\alpha_N)^M\,d\zeta}{(z-\zeta)Q(\zeta)}\right|\leqq\frac{1}{2\pi}\cdot\pi a_2^{m/m_N}\cdot 2^{m-m_N}a_1^{-m}\cdot 2^{m_N-M+1}a_2^{-m+(mM/m_N)}=$$

$$=\left(\frac{1}{2}a_2^{m/m_N}\right)^M\left(\frac{2}{a_1a_2}\right)^m.$$

Next, since $|z|=2a$, and $|\alpha_\nu|\leqq a$ for all ν,

$$|Q(z)|\leqq\prod_{\nu=0}^{n-1}(2a+a)^{m_\nu}=(3a)^m.$$

Hence the final result is that

$$|P_{MN}(z)|\leqq\frac{(3a)^m}{M!}\cdot\left(\frac{1}{2}a_2^{m/m_N}\right)^M\left(\frac{2}{a_1a_2}\right)^m,$$

whence, by $M \geqq 0$ and $a_2 \leqq 1$,

$$|P_{MN}(z)| \leqq \left(\frac{6a}{a_1 a_2}\right)^m \quad \text{if} \quad |z| = 2a. \tag{5}$$

4. The Taylor coefficients of $P_{MN}(z)$ are given by

$$P_{MN}^{(l)} = \frac{1}{2\pi i}\int \frac{P_{MN}(z)\,dz}{z^{l+1}} \qquad (0 \leqq l \leqq m-1),$$

where the integration extends, say, over the circle

$$|z| = 2a.$$

The estimate (5) implies therefore that

$$|P_{MN}^{(l)}| \leqq \frac{1}{2\pi}\cdot 4\pi a \cdot \left(\frac{6a}{a_1 a_2}\right)^m \cdot \frac{1}{(2a)^{l+1}}.$$

Since $a \geqq 1$, we find that

$$|P_{MN}^{(l)}| \leqq 2^{-l}\left(\frac{6a}{a_1 a_2}\right)^m \quad \text{for all} \quad M, N, l. \tag{6}$$

It is obvious that this formula remains valid in the trivial case when $n = 1$. For then

$$P_{M0}^{(l)} = \frac{\binom{M}{l}(-\alpha_0)^{M-l}}{M!} = \frac{(-\alpha_0)^{M-l}}{M!(M-l)!},$$

so that, by $M \leqq m_0 - 1 < m$,

$$|P_{M0}^{(l)}| \leqq a^{M-l} \leqq 2^{-l}(6a)^m \qquad (0 \leqq l \leqq m-1).$$

5. The inequalities (6) will now be applied to the function

$$E(z) = \sum_{\nu=0}^{n-1}\sum_{\mu=0}^{m_\nu-1} A_{\mu\nu} z^\mu e^{\alpha_\nu z}$$

where the numbers m_ν, m, n, and α_ν have the same meaning as before, and where the coefficients $A_{\mu\nu}$ are arbitrary complex numbers not all zero. Let

$$A = \max_{\substack{0 \leqq \mu \leqq m_\nu \\ 0 \leqq \nu \leqq n}} |A_{\mu\nu}| > 0$$

be the maximum of the absolute values of these coefficients.

Evidently,

$$E^{(l)}(0) = \sum_{\nu=0}^{n-1}\sum_{\mu=0}^{m_\nu-1} A_{\mu\nu}\sum_{\lambda=0}^{l}\binom{l}{\lambda}\alpha_\nu^{l-\lambda}\left(\frac{d}{dz}\right)^\lambda z^\mu\Bigg|_{z=0} =$$

$$= \sum_{\nu=0}^{n-1}\sum_{\mu=0}^{m_\nu-1} A_{\mu\nu}\binom{l}{\mu}\alpha_\nu^{l-\mu}\mu! = \sum_{\nu=0}^{n-1}\sum_{\mu=0}^{m_\nu-1} A_{\mu\nu}\left(\frac{d}{dz}\right)^\mu z^l\Bigg|_{z=\alpha_\nu}.$$

We multiply this equation by the coefficient $P_{MN}^{(l)}$ of $P_{MN}(z)$ and add over the values of l from 0 to $m-1$. Then

$$\sum_{l=0}^{m-1} P_{MN}^{(l)} E^{(l)}(0) = \sum_{\nu=0}^{n-1} \sum_{\mu=0}^{m_\nu - 1} A_{\mu\nu} \sum_{l=0}^{m-1} P_{MN}^{(l)} \left(\frac{d}{dz}\right)^{\mu} z^l \Big|_{z=\alpha_\nu} = \sum_{\nu=0}^{n-1} \sum_{\mu=0}^{m_\nu - 1} A_{\mu\nu} P_{MN}^{(\mu)}(\alpha_\nu),$$

so that, by (4),

$$\sum_{l=0}^{m-1} P_{MN}^{(l)} E^{(l)}(0) = A_{MN} \qquad (0 \leqq M \leqq m_N - 1,\ 0 \leqq N \leqq n-1).$$

Here the formula (6) implies that, for all suffixes M and N,

$$\sum_{l=0}^{m-1} |P_{MN}^{(l)}| \leqq \left(\frac{6a}{a_1 a_2}\right)^m \sum_{l=0}^{\infty} 2^{-l} = 2\left(\frac{6a}{a_1 a_2}\right)^m.$$

Hence it follows that

$$\max_{0 \leqq l \leqq m-1} |E^{(l)}(0)| \geqq \frac{1}{2} A \left(\frac{6a}{a_1 a_2}\right)^{-m}. \tag{7}$$

6. Let now $\beta_0, \beta_1, \ldots, \beta_{s-1}$ be a second set of finitely many distinct complex numbers, and let $r_0, r_1, \ldots, r_{s-1}$ be an equal number of positive integers. Then put

$$E = \max_{\substack{0 \leqq \sigma \leqq s-1 \\ 0 \leqq \varrho \leqq r_\sigma - 1}} |E^{(\varrho)}(\beta_\sigma)|;$$

our aim will be to establish an upper bound for A in terms of E when the integer

$$r = \sum_{\sigma=0}^{s-1} r_\sigma$$

is sufficiently large.

For this purpose we introduce the following three constants in analogy to a, a_1 and a_2,

$$b = \max_{0 \leqq \sigma \leqq s-1} (|\beta_\sigma|, 1), \quad b_1 = \min_{0 \leqq S \leqq s-1} \prod_{\substack{\sigma=0 \\ \sigma \neq S}}^{s-1} |\beta_S - \beta_\sigma|^{r_\sigma / r},$$

$$b_2 = \min_{\substack{0 \leqq \sigma \leqq s-1 \\ 0 \leqq S \leqq s-1 \\ S \neq \sigma}} (|\beta_S - \beta_\sigma|^{r_S / r}, 1).$$

In the trivial case $s=1$ these constants have the values,

$$b = \max(|\beta_0|, 1), \quad b_1 = 1, \quad b_2 = 1.$$

They always satisfy the inequalities

$$b \geqq 1 \geqq b_2$$

and

$$\prod_{\substack{\sigma=0 \\ \sigma \neq S}}^{s-1} |\beta_S - \beta_\sigma|^{r_\sigma} \geqq b_1^r, \quad |\beta_S - \beta_\sigma|^{r_S} \geqq b_2^r \quad \text{if} \quad S \neq \sigma.$$

7. If R is any positive number, put

$$M(R) = \max_{|\zeta|=R} |E(\zeta)|.$$

Further let

$$m^* = \max_{0 \leqq \nu \leqq n-1} m_\nu$$

denote the largest of the integers m_ν.

Assume that

$$R \geqq 3b \geqq 3. \tag{8}$$

An upper bound for $M(R)$ follows easily from the definition of $E(z)$. Evidently,

$$M(R) \leqq \sum_{\nu=0}^{n-1} A(1+R+\ldots+R^{m^*-1})e^{aR}.$$

Here

$$1+R+\ldots+R^{m^*-1} \leqq R^{m^*} \sum_{l=1}^{\infty} 3^{-l} < R^{m^*}.$$

It follows therefore that, if R satisfies the inequality (8), then

$$M(R) \leqq nAR^{m^*} e^{aR}. \tag{9}$$

8. Denote by z any complex number of absolute value

$$|z| = 2b. \tag{10}$$

This implies, in particular, that z has at least the distance b from each of the numbers $\beta_0, \beta_1, \ldots, \beta_{s-1}$.

Denote by Γ the circle

$$|\zeta| = R$$

in the ζ-plane, and put

$$J = \frac{1}{2\pi i} \int_\Gamma \prod_{\sigma=0}^{s-1} \left(\frac{z-\beta_\sigma}{\zeta-\beta_\sigma}\right)^{r_\sigma} \frac{E(\zeta)}{\zeta - z}\, d\zeta.$$

An upper bound for J is obtained in the following way.

If ζ lies on Γ, by (8) and (10),

$$\left|\prod_{\sigma=0}^{s-1} \left(\frac{z-\beta_\sigma}{\zeta-\beta_\sigma}\right)^{r_\sigma}\right| \leqq \prod_{\sigma=0}^{s-1} \left(\frac{2b+b}{R-\frac{1}{3}R}\right)^{r_\sigma} = \left(\frac{9b}{2R}\right)^r, \quad \left|\frac{E(\zeta)}{\zeta-z}\right| \leqq \frac{M(R)}{R-\frac{2}{3}R} = \frac{3M(R)}{R}.$$

Hence

$$|J| \leqq \frac{1}{2\pi} \cdot 2\pi R \cdot \left(\frac{9b}{2R}\right)^r \cdot \frac{3M(R)}{R} = 3\left(\frac{9b}{2R}\right)^r M(R),$$

whence, by (9),

$$|J| \leqq 3nA\left(\frac{9b}{2R}\right)^r R^{m^*} e^{aR}. \tag{11}$$

5. The residue theorem allows to express J in yet a second way. For $0 \leqq S \leqq s-1$, denote by Γ_S the circle

$$|\zeta-\beta_S| = \frac{1}{2}\, b_2^{r/rs},$$

and by Γ_s the circle

$$|\zeta - z| = \frac{1}{2}\, b.$$

Evidently all these circles lie outside one another, and hence

(12) $$J = \sum_{S=0}^{s} \frac{1}{2\pi i} \int_{\Gamma_S} \prod_{\sigma=0}^{s-1} \left(\frac{z-\beta_\sigma}{\zeta-\beta_\sigma}\right)^{r_\sigma} \frac{E(\zeta)}{\zeta - z}\, d\zeta, \quad = \sum_{S=0}^{s} J_S \quad \text{say.}$$

First, it is obvious that

(13) $$|J_s| = \frac{1}{2\pi i} \int_{\Gamma_s} \prod_{\sigma=0}^{s-1} \left(\frac{z-\beta_\sigma}{\zeta-\beta_\sigma}\right)^{r_\sigma} \frac{E(\zeta)}{\zeta - z}\, d\zeta = E(z)$$

because the integrand has inside Γ_s only the simple pole $\zeta = z$.

Secondly, let $0 \leqq S \leqq s-1$. By substituting for $E(\zeta)$ its Taylor series in powers of $\zeta - \beta_S$, J_S takes the form

$$J_S = \frac{1}{2\pi i} \int_{\Gamma_S} \prod_{\sigma=0}^{s-1} \left(\frac{z-\beta_\sigma}{\zeta-\beta_\sigma}\right)^{r_\sigma} \sum_{\varrho=0}^{\infty} \frac{E^{(\varrho)}(\beta_S)}{\varrho!} (\zeta-\beta_S)^{\varrho} \frac{d\zeta}{\zeta - z} =$$

$$= \sum_{\varrho=0}^{r_S-1} \sum_{\sigma=0}^{s-1} (z-\beta_\sigma)^{r_\sigma} \cdot \frac{E^{(\varrho)}(\beta_S)}{\varrho!} \cdot \frac{1}{2\pi i} \int_{\Gamma_S} \prod_{\substack{\sigma=0 \\ \sigma \neq S}}^{s-1} (\zeta-\beta_\sigma)^{-r_\sigma} \cdot (\zeta-\beta_S)^{\varrho - r_S} \frac{d\zeta}{\zeta - z}.$$

For all the integrals with $\varrho \geqq r_S$ vanish because their integrands remain regular inside Γ_S. By means of this representation, an upper bound for I_S may now be obtained by a method similar to that in § 3.

Assume that ζ lies on the circle Γ_S. Then for all $\sigma \neq S$,

$$|\zeta - \beta_S| = \frac{1}{2}\, b_2^{r/rs} \leqq \frac{1}{2}\, |\beta_S - \beta_\sigma|,$$

hence

$$|\zeta-\beta_\sigma| = |(\zeta-\beta_S)+(\beta_S-\beta_\sigma)| \geqq \frac{1}{2}\, |\beta_S - \beta_\sigma|$$

and therefore

$$\left| \prod_{\substack{\sigma=0 \\ \sigma \neq S}}^{s-1} (\zeta-\beta_\sigma)^{-r_\sigma} \right| \leqq \prod_{\substack{\sigma=0 \\ \sigma \neq S}}^{s-1} \{2^{-1} |\beta_S - \beta_\sigma|\}^{-r_\sigma} \leqq 2^{r - r_S} b_1^{-r}.$$

Further, by $b \geqq 1 \geqq b_2$ and by (10),

$$|\zeta| \leqq |(\zeta-\beta_S)+\beta_S| \leqq \frac{1}{2}\, b_2^{r/rs} + b \leqq \frac{3}{2}\, b, \quad \text{hence} \quad |\zeta - z| \geqq |z| - |\zeta| \geqq \frac{1}{2}\, b.$$

It follows then that for all suffixes ϱ with $0 \leqq \varrho \leqq r_S - 1$,

$$\left| \frac{1}{2\pi i} \int_{\Gamma_S} \prod_{\substack{\sigma=0 \\ \sigma \neq S}}^{s-1} (\zeta - \beta_\sigma)^{-r_\sigma} \cdot (\zeta - \beta_S)^{\varrho - r_S} \frac{d\zeta}{\zeta - z} \right| \leqq \frac{1}{2\pi} \cdot \frac{2\pi b_2^{r/r_S}}{2} \cdot 2^{r - r_S} b_1^{-r} \left(\frac{1}{2} b_2^{r/r_S} \right)^{\varrho - r_S} \cdot \frac{2}{b} =$$

$$= \frac{1}{b} \cdot 2^{-\varrho} (b_2^{r/r_S})^{\varrho+1} \left(\frac{2}{b_1 b_2} \right)^r.$$

Further, by (10),

$$\left| \prod_{\sigma=0}^{s-1} (z - \beta_\sigma)^{r_\sigma} \right| \leqq \prod_{\sigma=0}^{s-1} (2b + b)^{r_\sigma} = (3b)^r.$$

Thus, by $b_2 \leqq 1 \leqq b$ and by the definition of E, we find that

$$|J_S| \leqq \sum_{\varrho=0}^{r_S - 1} (3b)^r \cdot \frac{E}{\varrho!} \cdot \frac{1}{b} 2^{-\varrho} (b_2^{r/r_S})^{\varrho+1} \left(\frac{2}{b_1 b_2} \right)^r \leqq \left(\sum_{\varrho=0}^{\infty} \frac{2^{-\varrho}}{\varrho!} \right) \cdot \left(\frac{6b}{b_1 b_2} \right)^r E.$$

Here

$$\sum_{\varrho=0}^{\infty} \frac{2^{-\varrho}}{\varrho!} = \sqrt{e} < 2,$$

so that

$$|J_S| \leqq 2 \left(\frac{6b}{b_1 b_2} \right)^r E \qquad (S = 0, 1, \ldots, s-1). \tag{14}$$

We finally substitute the explicit value of J_s from (13) and the estimates for J and J_s from (11) and (14) in the identity (12). Since

$$E(z) = J - \sum_{S=0}^{s-1} J_S,$$

we arrive at the result,

$$|E(z)| \leqq 3nA \left(\frac{9b}{2R} \right)^r R^{m^*} e^{aR} + 2s \left(\frac{6b}{b_1 b_2} \right)^r E \quad \text{if} \quad |z| = 2b. \tag{15}$$

10. The derivatives of $E(z)$ at $z=0$ can be written as

$$E^{(l)}(0) = \frac{l!}{2\pi i} \int \frac{E(z)}{z^{l+1}} dz,$$

where, similarly as in §4, the integration is over the circle

$$|z| = 2b.$$

It follows that

$$|E^{(l)}(0)| \leqq \frac{l!}{2\pi} \cdot 4\pi b \cdot (2b)^{-l-1} \cdot \max_{|z|=2b} |E(z)|,$$

whence, by (15),

$$|E^{(l)}(0)| \leqq \frac{l!}{(2b)^l} \left\{ 3nA \left(\frac{9b}{2R} \right)^r R^{m^*} e^{aR} + 2s \left(\frac{6b}{b_1 b_2} \right)^r E \right\}. \tag{16}$$

We are interested only in the values of l with

$$0 \leqq l \leqq m-1.$$

Now

$$\frac{l!\,(2b)^{-l}}{(l-1)!\,(2b)^{-(l-1)}} = \frac{l}{2b} \begin{cases} \leqq 1 & \text{if} \quad 0 \leqq l \leqq 2b, \\ > 1 & \text{if} \quad l > 2b. \end{cases}$$

Hence, as l runs over the successive values 0, 1, 2, ..., the quotient

$$\frac{l!}{(2b)^l}$$

first decreases and then starts increasing. This evidently means that

$$\max_{0 \leqq l \leqq m-1} \frac{l!}{(2b)^l} = \max\left(1, \frac{(m-1)!}{(2b)^{m-1}}\right).$$

The inequality (16) implies therefore that

$$(17) \qquad \max_{0 \leqq l \leqq m-1} |E^{(l)}(0)| \leqq \max\left(1, \frac{(m-1)!}{(2b)^{m-1}}\right) \left\{3nA\left(\frac{9b}{2R}\right)^r R^{m^*} e^{aR} + 2s\left(\frac{6b}{b_1 b_2}\right)^r E\right\}.$$

11. In this inequality we choose now

$$R = \frac{m}{a}.$$

This is in agreement with the previous assumption that

$$(8) \qquad R \geqq 3b$$

provided

$$m \geqq 3ab.$$

Instead of this inequality we impose on m the stronger condition

$$(18) \qquad m \geqq 6ab.$$

It is then possible to show that

$$(m-1)! \geqq (2b)^{m-1}.$$

For, since $a \geqq 1$,

$$m \geqq 6b.$$

Consider now the sequence $\{c_k\}$ where

$$c_k = \frac{3^k k!}{k^k} \qquad (k = 1, 2, 3, \ldots).$$

Then

$$\frac{c_{k+1}}{c_k} = \frac{3}{\left(1+\frac{1}{k}\right)^k} > 1,$$

because, as is well-known, $\left\{\left(1+\frac{1}{k}\right)^k\right\}$ is an increasing sequence of limit $e<3$. Since $c_1=3$, it follows that $c_k \geqq 3$ for all k, and hence that

$$k! \geqq 3k^k 3^{-k},$$

whence

$$(m-1)! = \frac{m!}{m} \geqq \frac{3}{m} m^m 3^{-m} = \left(\frac{m}{3}\right)^{m-1} \geqq (2b)^{m-1},$$

as asserted.

On account of this inequality, the formula (17) takes the form,

$$\max_{0 \leqq l \leqq m-1} |E^{(l)}(0)| \leqq \frac{(m-1)!}{(2b)^{m-1}} \left\{3nA\left(\frac{9b}{2R}\right)^r R^{m^*} e^{aR} + 2s\left(\frac{6b}{b_1 b_2}\right)^r E\right\}.$$

We combine this result with the lower bound (7). This gives

$$\frac{1}{2} A\left(\frac{6a}{a_1 a_2}\right)^{-m} \leqq \frac{(m-1)!}{(2b)^{m-1}} \left\{3nA\left(\frac{9b}{2R}\right)^r R^{m^*} e^{aR} + 2s\left(\frac{6b}{b_1 b_2}\right)^r E\right\},$$

or, on replacing R by its value m/a,

$$\frac{1}{2} A\left(\frac{6a}{a_1 a_2}\right)^{-m} \leqq \frac{(m-1)!}{(2b)^{m-1}} \left\{3nA\left(\frac{9ab}{2m}\right)^r \left(\frac{m}{a}\right)^{m^*} e^m + 2s\left(\frac{6b}{b_1 b_2}\right)^r E\right\}. \tag{19}$$

12. In order to apply the last relation, let us assume that

$$\frac{1}{4}\left(\frac{6a}{a_1 a_2}\right)^{-m} \geqq \frac{(m-1)!}{(2b)^{m-1}} \cdot 3n\left(\frac{9ab}{2m}\right)^r \left(\frac{m}{a}\right)^{m^*} e^m; \tag{20}$$

it follows then immediately from (19) that

$$\frac{1}{4} A\left(\frac{6a}{a_1 a_2}\right)^{-m} \leqq \frac{(m-1)!}{(2b)^{m-1}} \cdot 2s\left(\frac{6b}{b_1 b_2}\right)^r E. \tag{21}$$

From these two formulas an important property of $E(z)$ will be deduced.

It is convenient to replace the assumption (20) by a stronger one which has a simpler form. For this purpose we use the well-known inequality

$$(m-1)! \leqq e m^{m-\frac{1}{2}} e^{-m}.$$

It shows that (20) is certainly satisfied if

$$\frac{1}{4}\left(\frac{6a}{a_1 a_2}\right)^{-m} \geqq \frac{e m^{m-\frac{1}{2}}}{(2b)^{m-1}} \cdot 3n\left(\frac{9ab}{2m}\right)^r \left(\frac{m}{a}\right)^{m^*},$$

or, what is the same, if

$$m^{r-m-m^*} \geqq \frac{24ebn}{\sqrt{m}}\left(\frac{6a}{a_1 a_2}\right)^m (2b)^{-m}\left(\frac{9ab}{2}\right)^r a^{-m^*}. \tag{22}$$

To simplify further, put

$$r = m + m^* + t$$

where t is a positive integer which will be fixed later. Then (22) becomes

$$m^t \geqq \frac{24ebn}{\sqrt{m}}\left(\frac{6a}{a_1 a_2}\right)^m (2b)^{-m}\left(\frac{9ab}{2}\right)^{m+m^*+t} a^{-m^*}$$

or

$$\left(\frac{2m}{9ab}\right)^t \geqq \frac{24ebn}{\sqrt{m}}\left(\frac{27a^2}{2a_1 a_2}\right)^m \left(\frac{9b}{2}\right)^{m^*}. \tag{23}$$

Here, by (18),

$$1 \leqq b \leqq \frac{m}{6},$$

while, trivially,

$$n \leqq m$$

by the definition of m and n in §1. Hence

$$\frac{24ebn}{\sqrt{m}} \leqq \frac{24e\cdot\frac{m}{6}\cdot m}{\sqrt{m}} = 4em^{3/2}.$$

Next, the expression

$$(4em^{3/2})^{1/m}$$

is easily seen to be a decreasing function of m, and m can by (18) not be smaller than 6; further

$$(4e\cdot 6^{3/2})^{1/6} < \frac{7}{3}.$$

It follows that

$$\frac{24ebn}{\sqrt{m}} < \left(\frac{7}{3}\right)^m.$$

Hence the inequality (23) is certainly satisfied if

$$\left(\frac{2m}{9ab}\right)^t \geqq \left(\frac{63a^2}{2a_1 a_2}\right)^m \left(\frac{9b}{2}\right)^{m^*}. \tag{24}$$

We assume from now that this inequality holds. Then the inequality (20) and hence also the inequality (21) are likewise true.

13. The inequality (21) is equivalent to

$$A \leqq 8s\frac{(m-1)!}{(2b)^{m-1}}\left(\frac{6a}{a_1 a_2}\right)^m \left(\frac{6b}{b_1 b_2}\right)^r E.$$

This relation will now be simplified in a similar manner as was (20). Again

$$(m-1)! \leqq em^{m-\frac{1}{2}}e^{-m}$$

and

$$r = m + m^* + t.$$

The inequality for A implies then that

$$A \leqq \frac{8s \cdot 2b \cdot e}{\sqrt{m}} \frac{m^m e^{-m}}{(2b)^m} \left(\frac{6a}{a_1 a_2}\right)^m \left(\frac{6b}{b_1 b_2}\right)^{m+m^*+t} E$$

or

$$A \leqq \frac{16ebs}{\sqrt{m}} \left(\frac{18am}{e \cdot a_1 a_2 b_1 b_2}\right)^m \left(\frac{6b}{b_1 b_2}\right)^{m^*+t} E. \tag{25}$$

In order to simplify this formula we use the trivial inequality

$$s \leqq r = m + m^* + t$$

which follows from the definition of r and s in §6. Therefore, by the theorem on the arithmetical and geometrical means,

$$s \leqq 2\sqrt{m} \cdot \sqrt{m^* + t},$$

and so, by $b \leqq \frac{m}{6}$,

$$\frac{16ebs}{\sqrt{m}} \leqq 16e \cdot \frac{m}{6} \cdot \frac{2\sqrt{m}}{\sqrt{m}} \cdot \sqrt{m^* + t}.$$

Here, similarly as above, by $m \geqq 6$,

$$\left(\frac{16em}{3}\right)^{1/m} \leqq \left(\frac{16e \cdot 6}{3}\right)^{1/6} < \frac{7}{3},$$

whence

$$\frac{16em}{3} \left(\frac{18am}{e \cdot a_1 a_2 b_1 b_2}\right)^m \leqq \left(\frac{42am}{e \cdot a_1 a_2 b_1 b_2}\right)^m \leqq \left(\frac{16am}{a_1 a_2 b_1 b_2}\right)^m.$$

Next, both m^* and t are positive integers, thus

$$m^* + t \geqq 2.$$

Since the function $x^{1/x}$ is increasing for $x < e$ and decreasing for $x > e$, it follows that

$$(m^* + t)^{\frac{1}{m^*+t}} \leqq \max(2^{1/2}, 3^{1/3}) < \left(\frac{4}{3}\right)^2$$

and hence that

$$\sqrt{m^* + t} \left(\frac{6b}{b_1 b_2}\right)^{m^*+t} \leqq \left(\frac{8b}{b_1 b_2}\right)^{m^*+t}.$$

We have then proved that if t satisfies the inequality (24), then A has the upper bound

$$A \leqq \left(\frac{16am}{a_1 a_2 b_1 b_2}\right)^m \left(\frac{8b}{b_1 b_2}\right)^{m^*+t} E \tag{26}$$

in terms of E.

We express this result in the form of a theorem.

THEOREM 1. *Let* $m_0, m_1, \ldots, m_{n-1}, n, r_0, r_1, \ldots, r_{s-1}, s$ *be positive integers, and let*

$$m = \sum_{\nu=0}^{n-1} m_\nu, \quad m^* = \max_{0 \leqq \nu \leqq n-1} m_\nu, \quad r = \sum_{\sigma=0}^{s-1} r_\sigma.$$

Let $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ *and* $\beta_0, \beta_1, \ldots, \beta_{s-1}$ *be* n *and* s *distinct complex numbers, respectively, and let*

$$a = \max_{0 \leqq \nu \leqq n-1} (|\alpha_\nu|, 1), \qquad b = \max_{0 \leqq \sigma \leqq s-1} (|\beta_\sigma|, 1),$$

$$a_1 = \min_{0 \leqq N \leqq n-1} \prod_{\substack{\nu=0 \\ \nu \neq N}}^{n-1} |\alpha_N - \alpha_\nu|^{m_\nu/m}, \qquad b_1 = \min_{0 \leqq S \leqq s-1} \prod_{\substack{\sigma=0 \\ \sigma \neq S}}^{s-1} |\beta_S - \beta_\sigma|^{r_\sigma/r},$$

$$a_2 = \min_{\substack{0 \leqq \nu \leqq n-1 \\ 0 \leqq N \leqq n-1 \\ N \neq \nu}} (|\alpha_N - \alpha_\nu|^{m_N/m}, 1), \qquad b_2 = \min_{\substack{0 \leqq \sigma \leqq s-1 \\ 0 \leqq S \leqq s-1 \\ S \neq \sigma}} (|\beta_S - \beta_\sigma|^{r_S/r}, 1).$$

Denote by

$$A_{\mu\nu} \qquad \begin{pmatrix} \mu = 0, 1, \ldots, m_\nu - 1 \\ \nu = 0, 1, \ldots, n-1 \end{pmatrix}$$

any set of m *complex numbers. Further put*

$$E(z) = \sum_{\nu=0}^{n-1} \sum_{\mu=0}^{m_\nu - 1} A_{\mu\nu} z^\mu e^{\alpha_\nu z}$$

and

$$A = \max_{\substack{0 \leqq \mu \leqq m_\nu - 1 \\ 0 \leqq \nu \leqq n-1}} (|A_{\mu\nu}|), \quad E = \max_{\substack{0 \leqq \varrho \leqq r_\sigma - 1 \\ 0 \leqq \sigma \leqq s-1}} (|E^{(\varrho)}(\beta_\sigma)|).$$

Assume, finally, that

$$m \geqq 6ab$$

and that further

$$r = m + m^* + t, \quad \text{where} \quad \left(\frac{2m}{9ab}\right)^t \geqq \left(\frac{63a^2}{2a_1 a_2}\right)^m \left(\frac{9b}{2}\right)^{m^*}.$$

Then

$$A \leqq \left(\frac{16am}{a_1 a_2 b_1 b_2}\right)^m \left(\frac{8b}{b_1 b_2}\right)^{m^*+t} E.$$

COROLLARY. *If* m *and* t *satisfy the hypothesis of the theorem, and if, in addition*

$$E^{(\varrho)}(\beta_\sigma) = 0 \qquad \begin{pmatrix} \varrho = 0, 1, \ldots, r_\sigma - 1 \\ \sigma = 0, 1, \ldots, s-1 \end{pmatrix},$$

then $A = 0$, *and* $E(z)$ *vanishes identically.*

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