

AN UNSOLVED PROBLEM ON THE POWERS OF $\frac{3}{2}$

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AN UNSOLVED PROBLEM ON THE POWERS OF $\frac{3}{2}$ *

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Let α be an arbitrary positive number. For every integer $n \geq 0$ we can write

$$\alpha\left(\frac{3}{2}\right)^n = g_n + r_n,$$

where

$$g_n = \left[\alpha\left(\frac{3}{2}\right)^n \right]$$

is the largest integer not greater than $\alpha\left(\frac{3}{2}\right)^n$, i.e. the integral part of $\alpha\left(\frac{3}{2}\right)^n$, and r_n is its fractional part and so satisfies the inequality

$$0 \leq r_n < 1.$$

We say that α is a *Z-number* if

$$(1) \quad 0 \leq r_n < \frac{1}{2} \quad \text{for all suffixes } n \geq 0.$$

Several years ago, a Japanese colleague proposed to me the problem whether such *Z-numbers* do in fact exist. I have not succeeded in solving this problem, but shall give here a number of incomplete results. In particular, it will be proved that *the set of all Z-numbers is at most countable*.

1

Assume that α is a *Z-number*. Evidently

$$g_{n+1} + r_{n+1} = \frac{3}{2}(g_n + r_n).$$

Here g_n and g_{n+1} are integers, while r_n and r_{n+1} lie in the interval

$$J = \left[0, \frac{1}{2}\right).$$

Hence one of the following two cases must hold.

(A) g_n is an even number, hence $\frac{3}{2}g_n$ is an integer. Since

$$0 \leq \frac{3}{2}r_n < \frac{3}{4},$$

necessarily

$$g_{n+1} = \frac{3}{2}g_n \quad \text{and} \quad r_{n+1} = \frac{3}{2}r_n.$$

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(B) g_n is an odd number and so both numbers $\frac{3}{2}g_n \mp \frac{1}{2}$ are integers. Since $\frac{3}{2}r_n + \frac{1}{2}$ cannot lie in J , we now must have

$$g_{n+1} = \frac{3}{2}g_n + \frac{1}{2} \text{ and } r_{n+1} = \frac{3}{2}r_n - \frac{1}{2}.$$

Put

$$\varepsilon_n = \begin{cases} 0 & \text{if } g_n \text{ is even,} \\ 1 & \text{if } g_n \text{ is odd.} \end{cases}$$

The two cases (A) and (B) can then be combined in the one formula

$$(2) \quad g_{n+1} = \frac{3}{2}g_n + \frac{1}{2}\varepsilon_n, \quad r_{n+1} = \frac{3}{2}r_n - \frac{1}{2}\varepsilon_n.$$

We also see that the case (A) can hold only if

$$0 \leq r_n < \frac{1}{3}$$

and case (B) if

$$\frac{1}{3} \leq r_n < \frac{1}{2}.$$

Hence ε_n may also be defined by

$$\varepsilon_n = \begin{cases} 0 & \text{if } 0 \leq r_n < \frac{1}{3}, \\ 1 & \text{if } \frac{1}{3} \leq r_n < \frac{1}{2}. \end{cases}$$

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From (2),

$$g_0 = -\frac{1}{3}\varepsilon_0 + \frac{2}{3}g_1, \quad g_1 = -\frac{1}{3}\varepsilon_1 + \frac{2}{3}g_2, \quad \dots, \quad g_{n-1} = -\frac{1}{3}\varepsilon_{n-1} + \frac{2}{3}g_n.$$

Since

$$g_0 + r_0 = \left(\frac{2}{3}\right)^n (g_n + r_n),$$

it follows from these equations that

$$(3) \quad g_0 = -\frac{1}{3}\{\varepsilon_0 + \frac{2}{3}\varepsilon_1 + \left(\frac{2}{3}\right)^2\varepsilon_2 + \dots + \left(\frac{2}{3}\right)^{n-1}\varepsilon_{n-1}\} + \left(\frac{2}{3}\right)^n g_n$$

and similarly also

$$(4) \quad r_0 = +\frac{1}{3}\{\varepsilon_0 + \frac{2}{3}\varepsilon_1 + \left(\frac{2}{3}\right)^2\varepsilon_2 + \dots + \left(\frac{2}{3}\right)^{n-1}\varepsilon_{n-1}\} + \left(\frac{2}{3}\right)^n r_n.$$

These equations can be generalised. For this purpose put

$$\alpha_0 = \alpha \text{ and } \alpha_m = \left(\frac{3}{2}\right)^m \alpha.$$

Then

$$\left(\frac{3}{2}\right)^n (g_m + r_m) = \left(\frac{3}{2}\right)^n \alpha_m = \left(\frac{3}{2}\right)^{m+n} \alpha = g_{m+n} + r_{m+n},$$

and it follows in analogy to (3) and (4) that for all suffixes m and n ,

$$(5) \quad g_m = -\frac{1}{3}\{\varepsilon_m + \frac{2}{3}\varepsilon_{m+1} + \left(\frac{2}{3}\right)^2\varepsilon_{m+2} + \dots + \left(\frac{2}{3}\right)^{n-1}\varepsilon_{m+n-1}\} + \left(\frac{2}{3}\right)^n g_{m+n}$$

and

$$(6) \quad r_m = +\frac{1}{3}\{\varepsilon_m + \frac{2}{3}\varepsilon_{m+1} + \left(\frac{2}{3}\right)^2\varepsilon_{m+2} + \dots + \left(\frac{2}{3}\right)^{n-1}\varepsilon_{m+n-1}\} + \left(\frac{2}{3}\right)^n r_{m+n}.$$

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The formula (6) for r_m immediately implies a convergent series for this number. For all r_{m+n} lie in the interval J , while the factor $(\frac{2}{3})^n$ tends to zero as n tends to infinity. It follows therefore that for all suffixes $m \geq 0$,

$$(7) \quad 3r_m = \varepsilon_m + \frac{2}{3}\varepsilon_{m+1} + (\frac{2}{3})^2\varepsilon_{m+2} + \dots$$

and in particular,

$$(8) \quad 3r_0 = \varepsilon_0 + \frac{2}{3}\varepsilon_1 + (\frac{2}{3})^2\varepsilon_2 + \dots$$

Here the convergence is in the sense of ordinary real analysis.

Consider next the formula (5) for g_m . The last term $(\frac{2}{3})^n g_{m+n}$ of this formula is a rational number the numerator of which is divisible by at least the n -th power of 2. In the so-called 2-adic analysis in the rational number field one considers numbers as small if they are divisible by a high power of 2 in the numerator, and as large if such a power of 2 occurs in the denominator. In this 2-adic sense the sequence of numbers $(\frac{2}{3})^n g_{m+n}$ tends to zero as n tends to infinity. We may therefore write

$$(9) \quad -3g_m = \varepsilon_m + \frac{2}{3}\varepsilon_{m+1} + (\frac{2}{3})^2\varepsilon_{m+2} + \dots \quad \text{in the 2-adic sense,}$$

and in particular,

$$(10) \quad -3g_0 = \varepsilon_0 + \frac{2}{3}\varepsilon_1 + (\frac{2}{3})^2\varepsilon_2 + \dots \quad \text{in the 2-adic sense.}$$

It is rather interesting that the same series converges in two different senses and to two different limits.

From this we can already deduce the fact *the set of all Z-numbers is at most countable*. For if the integer $g_0 \geq 0$ is given, then, by § 1, the corresponding sequence of integers $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$ is determined uniquely, and so, by (8), also the fractional part r_0 . We may express this result as follows.

(11) *For any given non-negative integer g_0 there exists at most one Z-number in the interval $[g_0, g_0+1)$, and this Z-number lies in fact in the first half $[g_0, g_0+\frac{1}{2})$ of this interval.*

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Much more can be said about the possible Z-numbers and their integral parts g_0 .

All the fractional parts r_m , where $r = 0, 1, 2, \dots$, lie by construction in the interval $J = [0, \frac{1}{2})$. This means by (7) that for every suffix m the inequality

$$(12) \quad \varepsilon_m + \frac{2}{3}\varepsilon_{m+1} + (\frac{2}{3})^2\varepsilon_{m+2} + \dots < \frac{3}{2}$$

is satisfied. In this set of inequalities each of the numbers $\varepsilon_m, \varepsilon_{m+1}, \varepsilon_{m+2}, \dots$ can assume only either of the two values 0 or 1.

It is then, firstly, immediately clear that *for no m simultaneously*

$$\varepsilon_m = \varepsilon_{m+1} = 1.$$

For this would imply that

$$\varepsilon_m + \frac{2}{3}\varepsilon_{m+1} + \left(\frac{2}{3}\right)^2\varepsilon_{m+2} + \dots \geq \frac{5}{3} > \frac{3}{2},$$

contrary to (12). Therefore

$$(13) \quad \text{if } m < n \text{ and } \varepsilon_m = \varepsilon_n = 1, \text{ then } n \geq m+2.$$

From the inequalities (12) one can deduce restrictions on those suffixes m for which simultaneously $\varepsilon_m = \varepsilon_{m+2} = 1, \varepsilon_{m+1} = 0$. We omit this discussion because no use will be made of the results so obtained.

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Denote from now on by

$$M = \{m_1, m_2, m_3, \dots\}, \quad \text{where } 0 \leq m_1 < m_2 < m_3 < \dots,$$

the set of all suffixes m for which $\varepsilon_m = 1$. Thus

$$\varepsilon_m = \begin{cases} 1 & \text{if } m \in M, \\ 0 & \text{if } m \notin M. \end{cases}$$

In other words, g_m is even or odd according as to whether m is, or is not, an element of M .

Further put

$$G_k = g_{m_k} \quad (k = 1, 2, 3, \dots),$$

so that all the G_k are odd.

On applying the equation (5) with

$$m = m_k \quad \text{and} \quad m+n = m_{k+1},$$

thus with

$$\varepsilon_m = 1, \quad \varepsilon_{m+1} = \varepsilon_{m+2} = \dots = \varepsilon_{m+n-1} = 0,$$

it follows that

$$G_k = -\frac{1}{3} + \left(\frac{2}{3}\right)^{m_{k+1}-m_k} G_{k+1},$$

hence that

$$(14) \quad G_{k+1} = \left(\frac{3}{2}\right)^{m_{k+1}-m_k-1} \frac{3G_k + 1}{2}.$$

This formula leads to the following algorithm connected with our problem.

We shall use the notation

$$2^a || H$$

to denote that H is divisible by 2^a , but not by 2^{a+1} .

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Put

$$(15) \quad a_k = m_{k+1} - m_k - 1, \quad H_k = \frac{3G_k + 1}{2}.$$

Then, by (14), the following properties hold.

For every $k \geq 1$,

$$(16) \quad G_k \text{ is odd; } H_k \text{ is even; } a_k \geq 1; \quad 2^{a_k} || H_k; \text{ and } G_{k+1} = \left(\frac{3}{2}\right)^{a_k} H_k \text{ is odd.}$$

Thus, starting with any odd integer G_1 , these formulae allow to determine successively the integers

$$H_1, a_1; G_2, H_2, a_2; G_3, H_3, a_3; \dots$$

If G_1 was the integral part of a Z -number, then this algorithm can be continued indefinitely. It thus provides a necessary (but not a sufficient) condition for G_1 to be the integral part of a Z -number.

By way of example, if we start with $G_1 = 13$, we obtain the following sequence of integers.

$G_1 = 13$	$H_1 = 20$	$a_1 = 2$
$G_2 = 45$	$H_2 = 68$	$a_2 = 2$
$G_3 = 153$	$H_3 = 230$	$a_3 = 1$
$G_4 = 345$	$H_4 = 518$	$a_4 = 1$
$G_5 = 777$	$H_5 = 1166$	$a_5 = 1$
$G_6 = 1749$	$H_6 = 2624$	$a_6 = 6$
$G_7 = 29889$	$H_7 = 44834$	$a_7 = 1$
$G_8 = 67251$	$H_8 = 100877$	

Since H_8 is odd, the algorithm breaks off, and there is no Z -number between 13 and 14.

In spite of much computer work, no integer G_1 is known for which the algorithm does not break off. It is thus highly problematical whether there do in fact exist Z -numbers.

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If the existence of Z -numbers is assumed, further properties of such numbers can be obtained.

Let us deal with the possible frequency of Z -numbers! We have already seen that there can be at most one Z -number in each interval between consecutive integers g and $g+1$ where $g \geq 0$. Thus, for $x > 0$, there are not more than $x+1$ Z -numbers between 0 and x . This estimate can now be replaced by a stronger one.

Let us first consider Z -numbers with *odd* integral parts, say with the integral part G_1 . Put

$$b_k = a_k + 1 \quad \text{and} \quad c_k = a_k - 1 \quad (k = 1, 2, 3, \dots),$$

so that by (16),

$$b_k \geq 2 \quad \text{and} \quad c_k \geq 0 \quad \text{for all } k.$$

By (15) and (16),

$$G_k = -\frac{1}{3} + \left(\frac{2}{3}\right)^{b_k} G_{k+1}.$$

On applying this equation repeatedly, we find that

$$(17) \quad G_1 = -\frac{1}{3} \left\{ 1 + \left(\frac{2}{3}\right)^{b_1} + \left(\frac{2}{3}\right)^{b_1+b_2} + \dots + \left(\frac{2}{3}\right)^{b_1+b_2+\dots+b_n} \right\} + \left(\frac{2}{3}\right)^{b_1+b_2+\dots+b_{n+1}} G_{n+2}.$$

Here

$$B_n = -\frac{1}{3} \left\{ 1 + \left(\frac{2}{3}\right)^{b_1} + \left(\frac{2}{3}\right)^{b_1+b_2} + \dots + \left(\frac{2}{3}\right)^{b_1+b_2+\dots+b_n} \right\}$$

is a rational number with an odd numerator and with a denominator which is a power of 3.

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Let now t be an arbitrarily large positive integer. For the given Z -number there exists just one suffix n such that

$$(18) \quad b_1 + b_2 + \dots + b_n \leq t < b_1 + b_2 + \dots + b_{n+1}.$$

There further is a unique integer s_n satisfying

$$1 \leq s_n \leq 2^t - 1$$

such that

$$B_n \equiv s_n \pmod{2^t},$$

i.e. that the numerator of $B_n - s_n$ is divisible by 2^t . It is then clear from (17) that also

$$(19) \quad G_1 \equiv s_n \pmod{2^t}.$$

The rational number B_n , and so also the integer s_n , depend only on t and on the ordered set of integers b_1, b_2, \dots, b_n . Denote by $T(t)$ the number of ordered sets of integers n, b_1, b_2, \dots, b_n which satisfy the left-hand inequality (18). This number $T(t)$ is then also the number of all residue classes $s_n \pmod{2^t}$ in which there can lie *odd* integral parts G_1 of Z -numbers.

One can easily obtain an upper bound for $T(t)$. The left-hand inequality (18) is equivalent to the inequality

$$c_1 + c_2 + \cdots + c_n \leq t - 2n;$$

hence $T(t)$ may also be defined as the number of ordered solutions n, c_1, c_2, \dots, c_n of this inequality where now c_1, c_2, \dots, c_n may run independently over all non-negative integers. For each separate value of n , this inequality has

$$\binom{[t-2n]+n}{n} = \binom{t-n}{n}$$

solutions, and hence, summing over n ,

$$T(t) = \binom{t-1}{1} + \binom{t-2}{2} + \binom{t-3}{3} + \cdots$$

where all terms after the $[\frac{t}{2}]$ -th vanish.

This formula may be written as

$$T(t) + 1 = \sum_{n=0}^t \binom{t-n}{n} = \sum_{n=0}^t \binom{n}{t-n}.$$

By the binomial theorem, it implies that $T(t) + 1$ is the coefficient of z^t in the power series in powers of z for

$$\sum_{n=0}^t \{z(1+z)\}^n = \frac{1 - \{z(1+z)\}^{t+1}}{1 - z(1+z)},$$

and hence $T(t) + 1$ is also the coefficient of z^t in the power series for

$$f(z) = \frac{1}{1-z-z^2}.$$

Put

$$A = \frac{1+\sqrt{5}}{2}, \quad B = \frac{1-\sqrt{5}}{2}, \quad \text{so that } A+B=1, \quad AB=-1, \quad A-B=\sqrt{5}.$$

Then

$$1-z-z^2 = (1-Az)(1-Bz) \quad \text{and} \quad f(z) = \frac{1}{\sqrt{5}} \left(\frac{A}{1-Az} - \frac{B}{1-Bz} \right).$$

On developing here $f(z)$ into a series in powers of z , it follows at once that

$$(20) \quad T(t) = \frac{1}{\sqrt{5}} \{A^{t+1} - B^{t+1}\} - 1.$$

Actually, $T(t) + 1$ is the $(t+1)$ -st term of the well known Fibonacci sequence.

Since trivially B^{t+1} has the limit 0 as t tends to infinity, and since further $A < \sqrt{5}$, it also follows from (20) that, for sufficiently large t ,

$$(21) \quad T(t) \leq \left(\frac{1 + \sqrt{5}}{2} \right)^t.$$

9

By the definition of $T(t)$, there are $T(t)$ distinct residue classes (mod 2^t) in which the integral part G_1 of a Z -number can lie when it is odd.

Consider next a Z -number $\alpha = g_0 + r_0$ with even integral part g_0 , say

$$2^m \parallel g_0.$$

Then

$$\alpha, \frac{3}{2}\alpha, \left(\frac{3}{2}\right)^2\alpha, \dots, \left(\frac{3}{2}\right)^m\alpha$$

likewise are Z -numbers, and they have the integral parts

$$g_0, \frac{3}{2}g_0, \left(\frac{3}{2}\right)^2g_0, \dots, \left(\frac{3}{2}\right)^mg_0,$$

respectively. Here $\left(\frac{3}{2}\right)^mg_0 = G_1$ say, is an odd integer divisible by 3^m , and

$$g_0 = \left(\frac{2}{3}\right)^m G_1, \quad \frac{3}{2}g_0 = \left(\frac{2}{3}\right)^{m-1} G_1, \dots, \quad \left(\frac{3}{2}\right)^m g_0 = G_1.$$

These $m+1$ products lie in the residue classes

$$(22) \quad \left(\frac{2}{3}\right)^\mu G_1 \pmod{2^t},$$

respectively, where μ runs over the successive values $\mu = m, m-1, m-2, \dots, 1, 0$. If $\mu \geq t$, then $\left(\frac{2}{3}\right)^\mu G_1$ lies in the residue class $\equiv 0 \pmod{2^t}$.

Thus to every *odd* residue class $G_1 \pmod{2^t}$ containing the integral part of a Z -number there correspond at most t *even* residue classes (22) in which there are likewise integral parts of Z -numbers.

(23) *This implies that there cannot be more than*

$$(t+1)T(t)$$

odd or even residue classes (mod 2^t) containing the integral part of a Z -number.

10

Trivially,

$$\frac{1 + \sqrt{5}}{2} < 2^{0.7}.$$

Thus, as soon as t is sufficiently large, it follows from (21) that there exist at most

$$2^{0.7 \cdot t-1}$$

odd or even residue classes (mod 2^t) in which there is the integral part of at least one Z -number.

Denote now by x a sufficiently large positive integer, and choose the integer t such that

$$2^t \leq x-1 < 2^{t+1}.$$

Then every residue class (mod 2^t) contains at most two integers $\leq x-1$. Hence there can be at most *two* Z -numbers not greater than x the integral parts of which lie in this residue class. By (23), the number of residue classes which need be considered is only

$$2^{0.7 \cdot t-1} < \frac{1}{2}x^{0.7}.$$

We obtain therefore the following result.

(24) *For sufficiently large x there are at most*

$$x^{0.7}$$

Z -numbers satisfying

$$0 \leq \alpha \leq x.$$

This paper dealt with the numbers α for which the fractional parts r_n defined in § 1 satisfied the inequalities

$$0 \leq r_n < \frac{1}{2} \quad (n = 0, 1, 2, \dots).$$

It is possible to establish a similar theory if all the r_n are assumed to lie in some other subinterval $[c, c + \frac{1}{2})$ of $[0, 1)$. It would be very interesting if a similar theory could be established for subintervals of smaller length, or perhaps even of arbitrarily small length.

Naturally, one can consider analogous problems for the products

$$\alpha \left(\frac{p}{q} \right)^n \quad (n = 0, 1, 2, \dots)$$

where α is again a positive number, and p and q are integers satisfying

$$p > q \geq 2, \quad (p, q) = 1.$$