Remarks on a Paper by W. Schwarz

K. MAHLER

*Department of Mathematics, Ohio State University,
Columbus, Ohio 43210*

Dedicated to L. J. Mordell on his eightieth birthday.

The author reports on old work of his on the transcendency of functions
satisfying functional equations like

$$F(z^2) = \frac{(1 - z)F(z) - z}{1 - z}.$$

He suggests a number of directions in which this work might possibly be
extended.

Recently, in a short paper, W. Schwarz [9] established a number of
results on irrationality and transcendency of values of the function

$$G_k(z) = \sum_{h=0}^{\infty} z^{k^h}/(1 - z^{k^h})$$

at certain rational points z; here k denotes a fixed integer $\geqq 2$. Schwarz
also considered the analogous problem for p-adic numbers. He gives in
his paper references to earlier work by S. Chowla [1], P. Erdös [2],
P. Erdös and E. G. Straus [3], and S. W. Golomb [4].

Surprisingly, Schwarz does not mention three papers of mine,
Mahler [6–8], of almost 40 years ago in which the problem of the trans-
cendency of functions like $G_k(z)$ was solved for all algebraic values of z,
and very general theorems were proved. In this note I shall therefore give
a short account of my old work and make suggestions for further
investigations.

1. Denote by

$$\mathbf{z} = (z_1, z_2, \ldots, z_n),$$

where $n \geqq 1$, a set of n independent complex variables, by E the $n \times n$ unit
matrix, by

$$\Omega = (o_{\alpha\beta}) \quad \text{and} \quad \Omega^k = (o_{\alpha\beta}^{(k)}) \qquad (k = 1, 2, 3, \ldots)$$

an $n \times n$ matrix with non-negative integral elements, and its kth power,
and by

$$\mathbf{z}' = \Omega^k \mathbf{z}$$

the transformation

$$z'_\alpha = \prod_{\beta=1}^{n} z_\beta^{o^{(k)}_{\alpha\beta}} \qquad \binom{\alpha = 1, 2, \ldots, n}{k = 1, 2, 3, \ldots}.$$

Let further $\rho_1, \rho_2, \ldots, \rho_n$ be the roots of the characteristic equation

$$|\Omega - \rho E| = 0,$$

numbered such that

$$|\rho_1| \geqq |\rho_2| \geqq \ldots \geqq |\rho_n|.$$

In addition, the matrix Ω is assumed to have the following properties.

(1): *The root ρ_1 is real, positive, and greater than 1.*
This condition unfortunately excludes important cases like that of the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and hence excludes Theta Functions from the consideration.

(2): *There is a neighbourhood U of the origin $\mathbf{0} = (0, 0, \ldots, 0)$ in complex n-dimensional $\mathbf{z}$-space such that*

$$\text{if } \mathbf{z} \in U \text{ and } k \to \infty, \text{ then } \Omega^k \mathbf{z} \to \mathbf{0}.$$

(3): *One can formulate a set of conditions (C) for the point $\mathbf{z}$ with the following property.*
If

$$F(\mathbf{z}) = \sum_{h_1=0}^{\infty} \ldots \sum_{h_n=0}^{\infty} A_{h_1 \ldots h_n} z_1^{h_1} \ldots z_n^{h_n} \not\equiv 0$$

is any power series convergent for $\mathbf{z} \in U$, and if $\mathbf{z}_0$ is any point in U which satisfies the conditions (C), then there exist arbitrarily large positive integers k such that

$$F(\Omega^k \mathbf{z}_0) \not\equiv 0.$$

My papers deal with two types of matrices Ω having such properties. In Mahler [6] I established the following result.

(A): *Let the characteristic equation $|\Omega - \rho E| = 0$ be irreducible, and let ρ_1 be greater than the absolute values of all the other roots $\rho_2, \ldots, \rho_n$. Then there exist n positive constants $\zeta_1, \ldots, \zeta_n$ depending only on Ω, with the following property.*
If $\mathbf{z}_0$ is any point with complex coordinates $z_{01}, \ldots, z_{0n}$ such that

$$z_{01} \neq 0, \ldots, z_{0n} \neq 0, \qquad \zeta_1 \log |z_{01}| + \ldots + \zeta_n \log |z_{0n}| < 0,$$

and if $F(\mathbf{z})$ is as in (3), then

$$F(\Omega^k \mathbf{z}_0) \neq 0 \text{ for all sufficiently large positive integers } k.$$

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The proof of (A) is elementary and depends only on classical properties of matrices.

In the paper (Mahler [7]), Ω was chosen differently. It was now assumed that all the roots $\rho_1, \ldots, \rho_n$ were equal to one and the same integer $\rho \geqq 2$ and that, more specifically, the matrix Ω had the form

$$\Omega = \rho E = \begin{pmatrix} \rho & 0 & \ldots & 0 \\ 0 & \rho & \ldots & 0 \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ 0 & 0 & \ldots & \rho \end{pmatrix}.$$

This case presented surprising arithmetic difficulties which were overcome by means of Siegel's generalization of Thue's theorem (Siegel [10]). The following theorem of the form (3) could then be established where the coordinates of the point $\mathbf{z}_0$ may now only be algebraic numbers.

(B): *Let $\mathbf{z}_0$ be a constant complex point with algebraic coordinates $z_{01}, \ldots, z_{0n}$ satisfying the inequalities*

$$0 < |z_{01}| < 1, \ldots, 0 < |z_{0n}| < 1.$$

Let there exist m algebraic numbers $Z_{01}, \ldots, Z_{0m}$, where $1 \leqq m \leqq n$, such that

$$Z_{01}^{e_1} \ldots Z_{0m}^{e_m} \neq 1$$

for every set of m integers $e_1, \ldots, e_m$ not all zero, and let further

$$z_{0h} = Z_{01}^{q_{1h}} \ldots Z_{0m}^{q_{mh}} \qquad (h = 1, 2, \ldots, n)$$

where the matrix (q_{gh}) possesses integral elements and the exact rank m. Assume, finally, that the function of m independent variables $Z_1, \ldots, Z_m$ defined by

$$G(Z_1, \ldots, Z_m) = F(z_1, \ldots, z_n), \text{ where } z_h = Z_1^{q_{1h}} \ldots Z_m^{q_{mh}}$$
$$(h = 1, 2, \ldots, n),$$

does not vanish identically.

Then there exists a sequence of arbitrarily large positive integers k for which

$$F(\Omega^k \mathbf{z}_0) \neq 0.$$

Since its solution would allow to generalize the results of my papers, I propose as a first subject for further research the following problem.

PROBLEM 1. *To establish results similar to (A) and (B) for more general classes of matrices Ω.*

There is a chance that a recent theorem by Hornich [5] may be useful in such an investigation. It should be possible to deal at least with matrices like

$$\begin{pmatrix} \Omega & 0 & \ldots & 0 \\ 0 & \Omega & \ldots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \ldots & \Omega \end{pmatrix},$$

where Ω denotes a matrix as in the result (A).

2. After additional restrictions were imposed on the function $F(\mathbf{z})$ in my papers (Mahler [6, 7]), it became possible to show the transcendency of the values of this function at certain algebraic points $\mathbf{z}_0$, and it was even possible to extend this to certain linear combinations of such function values.

These further restrictions are as follows.

(4): *The coefficients $A_{h_1 \ldots h_n}$ of $F(\mathbf{z})$ lie in K, an algebraic number field of finite degree of the rational field.*

(5): *The coordinates $z_{01}, \ldots, z_{0n}$ of $\mathbf{z}_0$ are algebraic numbers and therefore, without loss of generality, are elements of K.*

(6): *$F(\mathbf{z})$ satisfies a functional equation*

$$F(\Omega \mathbf{z}) = \frac{\sum_{l=0}^{m} a_l(\mathbf{z}) F(\mathbf{z})^l}{\sum_{l=0}^{m} b_l(\mathbf{z}) F(\mathbf{z})^l}.$$

Here $1 \leqq m < \rho_1$, and the $a_l(\mathbf{z})$ and $b_l(\mathbf{z})$ are polynomial in $z_1, \ldots, z_n$ with algebraic coefficients which, without loss of generality, lie in K.

(7): *At least one of the two polynomials*

$$A(\mathbf{z}, u) = \sum_{l=0}^{m} a_l(\mathbf{z}) u^l \quad and \quad B(\mathbf{z}, u) = \sum_{l=0}^{m} b_l(\mathbf{z}) u^l$$

is of the exact degree m in u. Denote by $\Delta(\mathbf{z})$ the resultant of $A(\mathbf{z}, u)$ and $B(\mathbf{z}, u)$, and assume that $A(\mathbf{z}, u)$ and $B(\mathbf{z}, u)$ are relatively prime, hence that $\Delta(\mathbf{z})$ is not identically zero. In the two exceptional cases when either $A(\mathbf{z}, u)$, or $B(\mathbf{z}, u)$, are independent of u, define $\Delta(\mathbf{z})$ instead by

$$\Delta(\mathbf{z}) = a_0(\mathbf{z}) b_m(\mathbf{z}) \quad and \quad \Delta(\mathbf{z}) = a_m(\mathbf{z}) b_0(\mathbf{z}),$$

respectively.

(8): *The coordinates $z_{01}, \ldots, z_{0n}$ of $\mathbf{z}_0$ are distinct from zero, and for every*
integer $k \geqq 0$ the point $\Omega^k \mathbf{z}_0$ lies in the convergence region of $F(\mathbf{z})$,
and

$$\Delta(\Omega^k \mathbf{z}_0) \neq 0.$$

The main result of my two papers is now as follows.

THEOREM 1. *If Ω, $F(\mathbf{z})$, and $\mathbf{z}_0$ satisfy the conditions (1)–(8), and if $F(\mathbf{z})$
is not an algebraic function of $z_1, \ldots, z_n$, then $F(\mathbf{z}_0)$ is not an algebraic
number.*

The proof of this theorem goes roughly as follows. Let p be a very large
positive integer. It is then possible to construct $p+1$ polynomials

$$A_0(\mathbf{z}), A_1(\mathbf{z}), \ldots, A_p(\mathbf{z})$$

not all identically zero, at most of degree p in each of the variables
$z_1, \ldots, z_n$, and with integral coefficients in K, such that the coefficients
$B_{h_1 \cdots h_n}$ in

$$E_p(\mathbf{z}) = \sum_{l=0}^{p} A_l(\mathbf{z}) F(\mathbf{z})^l = \sum_{h_1=0}^{\infty} \ldots \sum_{h_n=0}^{\infty} B_{h_1 \cdots h_n} z_1^{h_1} \ldots z_n^{h_n}$$

satisfy the equations

(9): $B_{h_1 \cdots h_n} = 0 \quad \text{if } h_1 + \ldots + h_n \leqq \tfrac{1}{2} p^{1+(1/n)}.$

Since $F(\mathbf{z})$ is a transcendental function, $E_p(\mathbf{z})$ *cannot vanish identically.*

Let also k be a large positive integer. By the functional equation,

$$F(\Omega^k \mathbf{z}) = \frac{\sum_{l=0}^{m^k} a_l^{(k)}(\mathbf{z}) F(\mathbf{z})^l}{\sum_{l=0}^{m^k} b_l^{(k)}(\mathbf{z}) F(\mathbf{z})^l},$$

where the $a_l^{(k)}(\mathbf{z})$ and $b_l^{(k)}(\mathbf{z})$ are polynomials with integral coefficients in K.
One easily can find majorants for these polynomials and their conjugates
relative to K. The same holds for the polynomials $B_l^{(k)}(\mathbf{z})$ in the
expressions

(10): $E_p^{(k)}(\mathbf{z}) = \left(\sum_{l=0}^{m^k} b_l^{(k)}(\mathbf{z}) F(\mathbf{z})^l \right)^p \cdot E_p(\Omega^k \mathbf{z}) = \sum_{l=0}^{pm^k} B_l^{(k)}(\mathbf{z}) F(\mathbf{z})^l.$

Substitute here for $\mathbf{z}$ the constant algebraic point $\mathbf{z}_0$, and assume that
$F(\mathbf{z}_0)$ is not a transcendental number, but is an algebraic number which,
without loss of generality, likewise lies in K.

It follows easily from the property (8) that

$$\sum_{l=0}^{m^k} b_l^{(k)}(\mathbf{z}_0) F(\mathbf{z}_0)^l \neq 0.$$

Further, by the property (3) applied to the function $E_p(\mathbf{z})$, there are arbitrarily large positive integers k for which

$$E_p(\Omega^k \mathbf{z}_0) \neq 0.$$

For each such value of k, the right-hand side of (10) is then distinct from zero. This right-hand side of (10) is a number in K, and the earlier majorants enable us to determine upper estimates for the absolute values of its conjugates relative to K. Therefore, by the usual norm procedure, one can find a positive constant c_1 independent of k and p, such that

$$\left| E_p^{(k)}(\mathbf{z}_0) \right| \geqq \exp\left(-c_1 \, p \rho_1^k \right).$$

On the other hand, one can deduce from (9) and from the majorants that

$$\left| E_p^{(k)}(\mathbf{z}_0) \right| \leqq \exp\left(-c_2 \, p^{1+(1/n)} \rho_1^k \right),$$

where c_2 is a second positive constant which is independent of k and p. Therefore, if p is sufficiently large, and k tends to infinity, a contradiction arises.

3. Let us, by way of example, apply Theorem 1 to the function $G_k(z)$ considered by Schwarz. In this case, $n = 1$; Ω is the 1×1 matrix (k); $G_k(z)$ satisfies the functional equation

$$F(\Omega z) = \frac{(1-z)F(z) - z}{1-z} \qquad (\Omega z = z^k);$$

and $\Delta(z) = (1-z)^2$. Further the series $G_k(z)$ converges for $|z| < 1$. The theorem implies therefore the following result which includes the theorems by Schwarz.

If z_0 is any real or complex algebraic number satisfying $0 < |z_0| < 1$, then the function value $G_k(z_0)$ is trancendental.

Since Theorem 1 is very general, one can deduce from it many other examples of this kind; for such, I refer to my two papers.

It has, however, some interest to mention one generalization of the property of $G_k(z)$ just stated.

Let $z_1, \ldots, z_n$ be again n independent complex variables, and let $a_1, \ldots, a_n$ be any n algebraic numbers distinct from zero. The function

$$F(\mathbf{z}) = \sum_{h=1}^{n} a_h G_k(z_h)$$

satisfies the functional equation

$$F(\Omega z) = F(\mathbf{z}) - \sum_{h=1}^{n} a_h z_h (1-z_h)^{-1}$$

where Ω now denotes the $n \times n$ matrix

$$\Omega = \begin{pmatrix} k & 0 & \ldots & 0 \\ 0 & k & \ldots & 0 \\ . & . & & . \\ . & . & & . \\ 0 & 0 & \ldots & k \end{pmatrix}.$$

We are thus now in the case studied in (B), and there is no difficulty in deducing from Theorem 1 the following result.

Let $z_{01}, \ldots, z_{0n}$ be n real or complex algebraic numbers such that

$$0 < |z_{01}| < 1, \ldots, 0 < |z_{0n}| < 1.$$

Let further

$$z_{01}^{e_1} \ldots z_{0n}^{e_n} \neq 1$$

for all integers $e_1, \ldots, e_n$ not all zero. Then, if $a_1, \ldots, a_n$ are any algebraic numbers distinct from zero, the expression

$$\sum_{h=1}^{n} a_h G_k(z_{0h})$$

is a transcendental number.

It would, in fact, suffice to assume that none of the quotients

$$\frac{\log z_{0h}}{\log z_{0j}},$$

where $h \neq j$, is an integral power of k.

4. For a more restricted type of functional equation, Theorem 1 can be replaced by a much stronger result, as I showed in my paper (Mahler [8]).

Let the matrix Ω still be as in (A); thus the equation $|\Omega - \rho E| = 0$ is irreducible, and its root $\rho_1 > 1$ is greater than the absolute values of all the other roots. Further let $\zeta_1, \ldots, \zeta_n$ be the same positive numbers as in (A), Section 1.

A rather involved proof leads then to the following result.

THEOREM 2. Let

$$a_1 > 0, \ldots, a_m > 0$$

be m algebraic constants, and let

$$b_1(\mathbf{z}), \ldots, b_m(\mathbf{z})$$

be m rational functions of $z_1, \ldots, z_n$ with algebraic coefficients such that

$$b_1(\mathbf{z}) = \ldots = b_m(\mathbf{z}) = 0 \quad \text{if } \mathbf{z} = \mathbf{0} = (0, \ldots, 0).$$

Let

$$f_\mu(\mathbf{z}) = \sum_{h_1=0}^{\infty} \cdots \sum_{h_n=0}^{\infty} f_{h_1 \cdots h_n}^{(\mu)} z_1^{h_1} \cdots z_n^{h_n} \qquad (\mu = 1, 2, \ldots, m)$$

be m power series with algebraic coefficients which satisfy the functional equations

$$f_\mu(\mathbf{z}) = a_\mu f_\mu(\Omega \mathbf{z}) + b_\mu(\mathbf{z}) \qquad (\mu = 1, 2, \ldots, m),$$

and converge in a neighbourhood of $z = 0$. Assume further that these functions are algebraically independent over the field of rational functions of $z_1, \ldots, z_n$ with complex coefficients.

Next let $\mathbf{z}_0$ be a complex point with algebraic coordinates $z_{01}, \ldots, z_{0n}$ satisfying

$$z_{01} \neq 0, \ldots, z_{0n} \neq 0, \quad \zeta_1 \log |z_{01}| + \ldots + \zeta_n \log |z_{0n}| < 0,$$

such that none of the points

$$\Omega^k \mathbf{z}_0 \qquad (k = 0, 1, 2, \ldots)$$

is a singular point of any of the m rational functions $b_1(\mathbf{z}), \ldots, b_m(\mathbf{z})$. Then the m function values

$$f_1(\mathbf{z}_0), \ldots, f_m(\mathbf{z}_0)$$

are algebraically independent over the field of rational numbers.

In the same paper, I also proved that the m functions $f_1(\mathbf{z}), \ldots, f_m(\mathbf{z})$ of Theorem 1 can only then be algebraically dependent over the field of rational functions of $z_1, \ldots, z_n$ if either at least one of these functions is rational, or if there are two distinct suffixes μ and v and two constants c_1 and c_2 such that

$$a_\mu = a_v \text{ and } f_\mu(\mathbf{z}) \equiv c_1 f_v(\mathbf{z}) + c_2 \text{ identically in } z_1, \ldots, z_n.$$

5. My paper (Mahler [8]) contained a number of applications of Theorem 1. Let us consider here an application of Schwarz's function $G_k(z)$.

If D is the differential operator

$$D = z \frac{\mathrm{d}}{\mathrm{d}z},$$

put

$$G_{kr}(z) = D^r G_k(z) = \sum_{h=0}^{\infty} D^r (z^{k^h} (1 - z^{k^h})^{-1}) \qquad (r = 0, 1, 2, \ldots).$$

Then

$$G_{k0}(z) = G_k(z),$$

and, for all r, $G_{kr}(z)$ satisfies the functional equation

$$G_{kr}(z) = D^r \{ z(1-z)^{-1} \} + k^r G_{kr}(z^k) \qquad (r = 0, 1, 2, \ldots).$$

Thus, with $\mu = r+1$ and for any positive integer m,

$$a_\mu = k^{\mu-1} \quad \text{and} \quad b_\mu(z) = D^{\mu-1}\{z(1-z)^{-1}\} \qquad (\mu = 1, 2, \ldots, m).$$

It is known that the function $G_k(z)$ cannot be continued beyond the unit circle. The same is therefore true for the functions $G_{k,\mu-1}(z)$; these functions are then certainly not rational. Also the constants a_μ are all distinct. Hence the result stated above implies that the functions

$$G_{k0}(z), G_{k1}(z), \ldots, G_{km}(z)$$

are algebraically independent over the field of rational functions of z. Theorem 2 leads then immediately to the following result.

If z_0 is any real or complex algebraic number satisfying $0 < |z_0| < 1$, and if m is any positive integer, then the m function values

$$G_{k0}(z_0), G_{k1}(z_0), \ldots, G_{k,m-1}(z_0)$$

are algebraically independent over the field of rational numbers.

From this result it is easily deduced that also the values of the derivatives

$$G_k(z_0), G_k'(z_0), \ldots, G_k^{(m-1)}(z_0)$$

are algebraically independent over the field of rational numbers. This property again implies in particular that $G_k(z)$, as function of z, does *not* satisfy any algebraic differential equation.

6. The methods of my three papers are quite general, and I believe that they can be further generalized. In this direction, I suggest an investigation of the following problem.

PROBLEM 2. *Let $\mathbf{z} = (z_1, \ldots, z_n)$ have the same meaning as before, and let*

$$\{\Omega\} = \{\Omega_1, \Omega_2, \Omega_3, \ldots\}$$

be an infinite sequence of matrices of the kinds considered in Section 1. Let further

$$F_r(\mathbf{z}) = \sum_{h_1=0}^{\infty} \ldots \sum_{h_n=0}^{\infty} F_{r, h_1 \cdots h_n} z_1^{h_1} \ldots z_n^{h_n} \qquad (r = 1, 2, 3, \ldots),$$

be an infinite sequence of power series with coefficients in the same algebraic number field K of finite degree over the rational number field, and let these series converge in a certain neighbourhood of the origin $\mathbf{z} = 0$. Assume, that for each suffix r, $F_r(\mathbf{z})$ satisfies a recursive formula

$$F_r(\mathbf{z}) = a_r(\mathbf{z})F_{r+1}(\Omega_r \mathbf{z}) + b_r(\mathbf{z}) \qquad (r = 1, 2, 3, \ldots),$$

where the $a_r(\mathbf{z})$ and $b_r(\mathbf{z})$ are rational functions of $z_1, \ldots, z_n$ with coefficients in K.

To obtain results as to the transcendency of $F_1(\mathbf{z}_0)$ when $\mathbf{z}_0$ is a point with algebraic coordinates $z_{01}, \ldots, z_{0n}$ suitably restricted.

Particularly promising seems to be the case where the matrix sequence (Ω) contains only finitely many distinct matrices, but is not necessarily periodic.

More important, but also probably much more difficult, would be a solution of the following problem, which would generalize my results in another direction.

PROBLEM 3. *Let* $\mathbf{z} = (z_1, \ldots, z_n)$ *and* Ω *be as before, and let*

$$F(\mathbf{z}) = \sum_{h_1=0}^{\infty} \cdots \sum_{h_n=0}^{\infty} F_{h_1 \cdots h_n} z_1^{h_1} \cdots z_n^{h_n}$$

be again a transcendental power series with algebraic coefficients which converges in a certain neighbourhood U of $\mathbf{z} = \mathbf{0}$. *Assume that* $F(\mathbf{z})$ *satisfies an algebraic functional equation of the form*

(P): $$P(F(\mathbf{z}), F(\Omega\mathbf{z}), \mathbf{z}) = 0,$$

where $P(u, v, \mathbf{z}) \neq 0$ *is a polynomial in* $u, v, z_1, \ldots, z_n$ *with algebraic coefficients.*

To investigate the transcendency of function values $F(\mathbf{z}_0)$ *where* $\mathbf{z}_0$ *is an algebraic point in U satisfying suitable further restrictions.*

The functional equation (P) is more general than the equations considered in my Theorems 1 and 2. As a consequence, the methods of my three papers do not seem to carry over, and new ideas will be necessary. A solution of Problem 3, even a partial one, would lead to many interesting applications. I mention in particular the connection to the theory of modular functions.

For let $n = 1$, $\Omega = (k)$ where $k \geqq 2$ is a fixed integer, and let

$$F(z) = j\left(\frac{\log z}{2\pi i}\right) - z^{-1},$$

where $j(\omega)$ is Weber's modular function of level 1. By the transformation theory of $j(\omega)$, $F(z)$ satisfies an algebraic functional equation (P). However, a decision as to whether $F(z_0)$ is transcendental for algebraic z_0, where $0 < |z_0| < 1$, does not seem to be easy.

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