

A Generalization of Farey Sequences: Some Exploration Via the Computer

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A generalization of Farey sequences for higher dimensions is considered, and numerical results obtained via the computer for quadratic Farey sequences are presented.

The Farey sequence of order n on the interval $[a, b]$ consists of all rational points in this interval whose denominators do not exceed n . These sequences possess many interesting properties, and they play an important role in the approximation of irrationalities by rationals [1].

The Farey sequence of order n , say on $[0, 1]$, can also be described as the sequence of all roots of the set of integral linear polynomials $ax - b$ where $0 \leq b \leq a \leq n$. Using this latter description, the Farey sequence admits an immediate generalization. Namely, the m -th degree Farey sequence of order n is the sequence of all real roots of the set of integral polynomials

$$a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0, \text{ where } |a_i| \leq n.$$

We have carried out extensive computations on the quadratic Farey sequences of orders 2–25. These computations were performed on an IBM 7094 using an efficient assembly language program. Table I gives the positive members of the quadratic Farey sequence of order 5 and some related data. The phenomena exhibited in this table are typical of the phenomena exhibited in all of the computed tables. The listed members of the sequence are arranged in order of increasing magnitude. The fourth column in the table gives six place decimal approximations to the points in the sequence, while the first three columns give the coefficients (leading coefficient first) of the primitive polynomials determining these points. The fifth column lists the determinant formed by the coefficients of the preceding polynomial, the associated polynomial and the succeeding

TABLE I

POLYNOMIAL	ROOT	DETERMINANT	COMMON	VALUE	SIGN	RESULTANT		
5	5	-1	0.170820	-0	0.170820	0.000000	-1	0
4	5	-1	0.175390	0	0.173049	0.014631	-1	1
3	5	-1	0.180460	0	0.177857	0.015642	-1	1
2	5	-1	0.186141	0	0.183216	0.016662	-1	1
1	5	-1	0.192582	0	0.189255	0.017689	-1	1
0	5	-1	0.200000	-1	0.196152	0.019100	-1	1
4	4	-1	0.207107	-1	0.203768	0.018841	-1	-4
1	-5	1	0.208712	1	0.207825	0.003558	1	1
3	4	-1	0.215250	1	0.212214	-0.016164	1	-3
2	-5	1	0.219224	1	0.216990	0.008965	1	2
2	4	-1	0.224745	1	0.222222	-0.012394	1	-2
3	-5	1	0.232408	1	0.227998	0.015541	1	3
1	4	-1	0.236068	-0	0.234436	-0.007383	1	-1
5	3	-1	0.238516	1	0.237405	0.005932	-1	-1
0	4	-1	0.250000	-2	0.243398	0.025838	-1	5
3	3	-1	0.263763	-1	0.257334	0.029335	-1	-3
1	-4	1	0.267949	-0	0.265564	0.007585	1	1
5	-5	1	0.276393	-1	0.271286	-0.011707	-1	1
2	3	-1	0.280776	-1	0.279241	-0.007532	1	-1
5	2	-1	0.289898	-1	0.285714	0.020328	-1	-2
2	-4	1	0.292893	1	0.290994	0.004019	1	1
1	3	-1	0.302776	-0	0.298438	-0.015738	1	-1
5	5	-2	0.306226	0	0.305159	0.008310	-1	-1
4	2	-1	0.309017	1	0.307220	0.007742	-1	-1
4	5	-2	0.318729	0	0.315100	0.026952	-1	4
0	3	-1	0.333333	-3	0.322876	0.031058	-1	4
5	4	-2	0.348331	1	0.344031	0.032092	-1	-5
2	5	-2	0.350781	-1	0.349460	0.007269	-1	2
5	1	-1	0.358258	1	0.353889	0.019393	-1	-5
2	2	-1	0.366025	0	0.361590	0.014380	-1	1
1	5	-2	0.372281	0	0.369924	0.013099	-1	1
1	-3	1	0.381966	1	0.375000	0.015625	1	1
3	4	-2	0.387426	-0	0.386001	-0.009206	1	-1
4	1	-1	0.390388	1	0.388593	0.006433	-1	-1
0	5	-2	0.400000	-1	0.395644	0.021350	-1	4
1	2	-1	0.414214	0	0.405125	0.025624	-1	-1
5	5	-3	0.421954	0	0.420133	0.016600	-1	-1
4	3	-2	0.425390	0	0.423361	0.012888	-1	-1
3	1	-1	0.434258	-1	0.428571	0.019213	-1	-1
1	-5	2	0.438447	1	0.436492	0.006872	1	1
4	5	-3	0.443000	-0	0.441518	-0.012753	1	-4
5	0	-1	0.447214	1	0.444444	0.011960	-1	-4
1	4	-2	0.449490	-1	0.448403	0.004605	-1	1
3	3	-2	0.457427	0	0.453768	0.020838	-1	-2
5	2	-2	0.463325	1	0.460582	0.017787	-1	-2
3	5	-3	0.468375	-1	0.466052	0.017047	-1	5
5	4	-3	0.471780	1	0.470169	0.012776	-1	-3
0	2	-1	0.500000	-5	0.477033	0.043691	-1	5
5	5	-4	0.524695	0	0.520656	0.041311	-1	-5
5	3	-3	0.530662	1	0.527362	0.027266	-1	-5
3	4	-3	0.535184	-1	0.532761	0.017208	-1	3
5	1	-2	0.540312	1	0.537592	0.015827	-1	-4
1	5	-3	0.541381	-1	0.540833	0.001682	-1	1
3	2	-2	0.548584	1	0.544727	0.020043	-1	-3
4	5	-4	0.554248	-1	0.552209	0.017727	-1	2
5	-1	-1	0.558257	1	0.555555	0.010658	-1	-5
1	3	-2	0.561553	-1	0.559816	0.005778	-1	1
4	3	-3	0.568729	0	0.566190	0.018834	-1	-2
3	0	-1	0.577350	1	0.571429	0.019770	-1	-2
5	4	-4	0.579796	0	0.579156	0.006076	-1	1
2	4	-3	0.581139	0	0.580323	0.003743	-1	1
1	-4	2	0.585786	1	0.582576	0.008961	1	1
3	5	-4	0.590667	-0	0.589454	-0.010476	1	-2
4	1	-2	0.593070	1	0.591633	0.007101	-1	-2
0	5	-3	0.600000	-1	0.596291	0.018068	-1	4
1	1	-1	0.618034	0	0.605551	0.027756	-1	-1
5	0	-2	0.632455	1	0.628667	0.023576	-1	-1
2	5	-4	0.637459	-1	0.635175	0.014637	-1	6
4	-1	-1	0.640388	1	0.638492	0.007016	-1	-2
5	3	-4	0.643398	0	0.642481	0.006636	-1	2
1	4	-3	0.645751	-1	0.644243	0.006686	-1	2
4	2	-3	0.651388	1	0.649000	0.017103	-1	-3

4	5	-5	0.655869	0	0.654016	0.017894	-1	4
0	3	-2	0.666667	-3	0.658312	0.024487	-1	4
5	4	-5	0.677033	0	0.674773	0.024318	-1	-5
5	1	-3	0.681025	1	0.678709	0.014765	-1	-5
2	3	-3	0.686141	-1	0.683190	0.016355	-1	3
5	-2	-1	0.689898	1	0.687868	0.008915	-1	-2
4	3	-4	0.693000	0	0.691868	0.006477	-1	3
1	-5	3	0.697224	1	0.694254	0.008461	1	3
1	5	-4	0.701562	-0	0.700000	-0.010410	1	-1
3	5	-5	0.703257	0	0.702562	0.005824	-1	-1
2	0	-1	0.707107	1	0.704159	0.008074	-1	-1
5	2	-4	0.716515	0	0.714286	0.019213	-1	1
3	2	-3	0.720759	-1	0.718246	0.012485	-1	1
5	-5	1	0.723607	1	0.721500	0.003351	-1	-1
4	4	-5	0.724745	-1	0.724533	-0.000908	-1	1
1	2	-2	0.732051	-1	0.726650	0.018172	-1	1
5	-1	-2	0.740312	1	0.737405	0.018428	-1	-2
5	3	-5	0.744031	0	0.742615	0.011597	-1	5
0	4	-3	0.750000	-2	0.745683	0.016687	-1	5
3	3	-4	0.758306	1	0.755427	0.021708	-1	-3
2	5	-5	0.765564	-1	0.762050	0.026796	-1	4
3	-1	-1	0.767592	1	0.766190	0.004811	-1	-1
5	0	-3	0.774597	0	0.772363	0.015462	-1	1
2	1	-2	0.780776	0	0.776739	0.013730	-1	1
3	4	-5	0.786300	0	0.784523	0.014399	-1	1
1	3	-3	0.791288	0	0.788017	0.012997	-1	1
0	5	-4	0.800000	-1	0.795831	0.020172	-1	1
4	3	-5	0.804248	0	0.802776	0.013878	-1	-4
4	-2	-1	0.809017	1	0.805778	0.013711	-1	-4
3	0	-2	0.816496	0	0.812971	0.014792	-1	1
5	2	-5	0.819804	0	0.818729	0.009136	-1	1
2	2	-3	0.822876	0	0.820852	0.010274	-1	1
1	4	-4	0.828427	-1	0.825742	0.014990	-1	1
5	-3	-1	0.838516	1	0.833333	0.027235	-1	-7
4	-1	-2	0.843070	0	0.840863	0.012306	-1	1
3	1	-3	0.847127	0	0.845154	0.009098	-1	1
2	3	-4	0.850781	0	0.849000	0.009532	-1	1
1	5	-5	0.854102	-1	0.852479	0.009419	-1	1
5	-2	-2	0.863325	1	0.858678	0.030237	-1	-11
4	0	-3	0.866025	0	0.864703	0.005301	-1	1
3	2	-4	0.868517	0	0.867295	0.008057	-1	1
2	4	-5	0.870829	-1	0.869694	0.008151	-1	1
5	-1	-3	0.881025	1	0.876024	0.038766	-1	-13
4	1	-4	0.882782	0	0.881917	0.006256	-1	1
3	3	-5	0.884437	-1	0.883622	0.004503	-1	1
5	0	-4	0.894427	1	0.889605	0.040634	-1	-11
4	2	-5	0.895644	-1	0.895043	0.005300	-1	1
5	1	-5	0.904987	1	0.900521	0.043854	-1	-5
0	1	-1	1.000000	-10	0.913553	0.082182	-1	5
5	-1	-5	1.104988	-0	1.095445	0.095445	-1	-5
5	-2	-4	1.116515	1	1.110469	0.053934	-1	-5
4	0	-5	1.118034	-1	1.117265	0.006010	-1	1
5	-3	-3	1.130662	1	1.124094	0.053715	-1	-11
4	-1	-4	1.132782	-0	1.131706	0.004620	-1	1
3	1	-5	1.135042	-1	1.133893	0.008017	-1	1
5	-4	-2	1.148331	1	1.141521	0.047940	-1	-13
4	-2	-3	1.151388	-0	1.149830	0.006444	-1	1
3	0	-4	1.154701	0	1.153010	0.010067	-1	1
2	2	-5	1.158312	-1	1.156466	0.010742	-1	1
5	-5	-1	1.170820	1	1.164581	0.041519	-1	-11
4	-3	-2	1.175390	-0	1.173049	0.012661	-1	1
3	-1	-3	1.180460	-0	1.177857	0.014663	-1	1
2	1	-4	1.186141	0	1.183216	0.016000	-1	1
1	3	-5	1.192582	-1	1.189255	0.016530	-1	1
4	-4	-1	1.207107	1	1.200000	0.039297	-1	-7
3	-2	-2	1.215250	-0	1.211032	0.021350	-1	1
5	-2	-5	1.219804	-0	1.218246	0.012425	-1	1
2	0	-3	1.224745	0	1.221405	0.013732	-1	1
1	2	-4	1.236068	-1	1.230139	0.025704	-1	1
5	-3	-4	1.243398	1	1.241037	0.021466	-1	-4
0	4	-5	1.250000	-2	1.245362	0.015497	-1	5
3	-3	-1	1.263763	1	1.257334	0.029335	-1	-3
5	-4	-3	1.271780	-0	1.269008	0.020668	-1	1
2	-1	-2	1.280776	-0	1.274659	0.022018	-1	1
3	0	-5	1.290994	0	1.287434	0.025867	-1	1
1	1	-3	1.302776	-1	1.294727	0.026675	-1	1
5	-5	-2	1.306226	1	1.305159	0.007368	-1	-1
4	-3	-3	1.318729	1	1.312250	0.045624	-1	4

0	3	-4	1.333333	-3	1.322876	0.030085	-1	4
5	-3	-5	1.344031	1	1.341641	0.024922	-1	-5
2	1	-5	1.350781	-1	1.346593	0.021378	-1	5
2	-2	-1	1.366025	1	1.356107	0.033346	-1	-2
5	-4	-4	1.379796	-1	1.376179	0.030426	-1	1
1	-5	5	1.381966	1	1.380199	-0.000414	1	1
3	-2	-3	1.387426	-1	1.396001	-0.009723	1	-1
4	-2	-5	1.395644	-0	1.392281	0.029181	-1	1
1	0	-2	1.414214	-1	1.400000	0.038906	-1	1
5	-5	-3	1.421954	-0	1.420133	0.016178	-1	-1
4	-5	-1	1.425390	-0	1.423361	0.012675	-1	-1
3	-5	1	1.434258	1	1.428571	0.016423	-1	-1
4	-3	-4	1.443000	0	1.440395	0.018321	-1	3
1	2	-5	1.449490	-1	1.445362	0.019626	-1	3
3	-3	-2	1.457427	1	1.453768	0.020528	-1	-2
3	-1	-5	1.468375	-1	1.463722	0.033269	-1	3
5	-4	-5	1.477033	1	1.473386	0.037078	-1	-5
0	2	-3	1.500000	-5	1.480625	0.036439	-1	5
5	-5	-4	1.524695	1	1.520656	0.041311	-1	-5
3	-2	-4	1.535184	-1	1.529016	0.041201	-1	4
3	-4	-1	1.548584	1	1.540833	0.038007	-1	-3
4	-3	-5	1.554248	-0	1.552209	0.015023	-1	2
1	1	-4	1.561553	-1	1.556466	0.017043	-1	2
4	-5	-2	1.568729	1	1.566190	0.018289	-1	-2
2	0	-5	1.581139	-1	1.574370	0.040807	-1	6
1	-1	-1	1.618034	1	1.590667	0.058312	-1	-1
0	3	-5	1.666667	-1	1.645751	0.061945	-1	1
2	-1	-4	1.686141	-0	1.679449	0.038348	-1	-2
4	-5	-3	1.693000	-0	1.690238	0.021566	-1	-2
2	-4	1	1.707107	1	1.696485	0.027163	-1	-2
3	-4	-2	1.720759	0	1.716515	0.025370	-1	1
4	-4	-5	1.724745	-0	1.723180	0.014963	-1	1
1	0	-3	1.732051	-1	1.726650	0.017473	-1	1
3	-3	-4	1.758306	-0	1.750000	0.062500	-1	-2
2	-3	-1	1.780776	1	1.766190	0.058897	-1	-2
1	1	-5	1.791288	-1	1.786300	0.021642	-1	1
4	-5	-4	1.804248	1	1.800000	0.039297	-1	-5
2	-2	-3	1.822876	-1	1.810910	0.057770	-1	2
3	-5	-1	1.847127	1	1.835782	0.066185	-1	-3
2	-1	-5	1.850781	-1	1.849000	0.007332	-1	1
3	-4	-3	1.868517	1	1.860147	0.058036	-1	-4
3	-3	-5	1.884437	-1	1.877015	0.060801	-1	3
4	-5	-5	1.905869	1	1.896231	0.095710	-1	-5
0	1	-2	2.000000	-7	1.914214	0.083473	-1	4
3	-4	-5	2.119633	-0	2.107275	0.107275	-1	-3
3	-5	-3	2.135042	1	2.126893	0.057988	-1	-3
2	-2	-5	2.158312	-1	2.145683	0.081195	-1	5
2	-3	-3	2.186141	-0	2.171165	0.083271	-1	-2
2	-4	-1	2.224745	1	2.203768	0.100519	-1	-2
1	0	-5	2.236068	-1	2.230139	0.025072	-1	1
3	-5	-4	2.257334	-0	2.250000	0.062500	-1	-4
2	-5	1	2.280776	1	2.264911	0.061342	-1	-4
1	-1	-3	2.302776	-1	2.290994	0.038059	-1	1
2	-3	-4	2.350781	-0	2.333333	0.109592	-1	-1
3	-5	-5	2.369924	-0	2.362050	0.070204	-1	-1
1	-2	-1	2.414214	1	2.380199	0.090530	-1	-1
0	2	-5	2.500000	-1	2.449490	0.099693	-1	1
1	-1	-4	2.561553	-0	2.541381	0.082762	-1	-1
2	-4	-3	2.581139	-0	2.573384	0.048191	-1	-1
1	-3	1	2.618034	1	2.590667	0.056971	-1	-1
2	-5	-1	2.686141	0	2.666667	0.110243	-1	1
1	-2	-2	2.732051	-0	2.703257	0.097483	-1	1
1	-1	-5	2.791288	-1	2.765564	0.114852	-1	1
2	-5	-2	2.850781	-1	2.825742	0.158746	-1	-5
2	-4	-5	2.870829	-0	2.861555	0.066620	-1	2
0	1	-3	3.000000	-4	2.896001	0.111012	-1	2
2	-5	-4	3.137459	1	3.121320	0.121320	-1	-2
1	-2	-4	3.236068	-1	3.173599	0.272177	-1	4
2	-5	-5	3.265564	-0	3.254994	0.084274	-1	-1
1	-3	-1	3.302776	-0	3.276984	0.090644	-1	-1
1	-4	2	3.414214	1	3.350781	0.174083	-1	-1
1	-2	-5	3.449490	-1	3.436492	0.060371	-1	1
1	-3	-2	3.561553	-1	3.500000	0.250000	-1	-2
1	-5	5	3.618034	1	3.581139	0.080536	-1	-1
1	-4	1	3.732051	0	3.696141	0.156152	-1	1
1	-3	-3	3.791288	0	3.765564	0.113997	-1	1
0	1	-4	4.000000	-2	3.828427	0.169260	-1	1
1	-3	-5	4.192582	-0	4.162278	0.162278	-1	-1

1	-4	-1	4.236068	-0	4.212214	0.103577	-1	-1
1	-5	3	4.302776	1	4.265564	0.129134	-1	-1
1	-4	-2	4.449490	-1	4.386001	0.304730	-1	3
1	-5	2	4.561553	1	4.500000	0.250000	-1	-2
1	-4	3	4.645751	-1	4.608495	0.194860	-1	2
1	-5	1	4.791288	1	4.712214	0.353277	-1	-3
1	-4	-4	4.828427	0	4.811738	0.092070	-1	1
0	1	-5	5.000000	-2	4.854102	0.144946	-1	1
1	-5	-1	5.192582	1	5.162278	0.162278	-1	-1
1	-5	-2	5.372281	-0	5.283882	0.497984	-1	
1	-5	-3	5.541381	-0	5.458040	0.499650	-1	1
1	-5	-4	5.701562	-0	5.622499	0.495180	-1	1
1	-5	-5	5.854102	-0	5.778719	0.495390	-1	1

polynomial. The sixth column lists a six place decimal approximation to the unique real number lying between the given point in the sequence and its predecessor at which the associated polynomials have, up to sign, the same value. The next column lists a decimal approximation to this common value, and the eighth column indicates whether this common value occurs with the same (+1) or opposite (-1) sign. The last column lists the resultant of the given polynomial and its predecessor.

As the tables show, the quadratic Farey sequences seem to possess some interesting properties. For example, one of the properties of the linear Farey sequences is that the determinant formed by the coefficients of the primitive polynomial associated with a given point and the coefficients of the preceding primitive polynomial is always 1. As far as our computations of the quadratic Farey sequences go, the associated determinants have always the values 1, -1, or 0, the only exceptions being at the rational points which belong to the linear Farey sequence of order n . Moreover, the absolute values of these determinants at the rational points vary with the size of the denominators. Table II lists the absolute values of the determinants (they actually are all negative) associated with the rational members of quadratic Farey sequences of orders 2-10 on the interval $[0,1]$.

The resultants of the successive pairs of polynomials in the computed quadratic Farey sequences also seem to follow a pattern; however, this pattern is more complex. As far as the computations go, roughly one half of these resultants have the values +1 or -1; there is, however, seemingly no simple rule by which these and the larger values are attained.

So far we have not been able to prove any of our observations about these generalized Farey sequences. However, we do feel that our numerical data might, in itself, be of interest.

REFERENCE

1. G. H. HARDY AND E. M. WRIGHT, "An Introduction to the Theory of Numbers," 4th ed., Clarendon, Oxford, 1965.