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Matematica. — *A remark on algebraic differential equations.*

Nota di KURT MAHLER, presentata (*) dal Socio B. SEGRE.

Riassunto. — Sia f una serie di potenze formale in un'indeterminata z , a coefficienti in un qualsiasi campo di caratteristica zero. Si dimostra che, se f soddisfa ad un'equazione differenziale algebrica (alle derivate ordinarie rispetto alla z) e coefficienti in un campo di caratteristica zero, allora f soddisfa di conseguenza ad un'equazione differenziale algebrica a coefficienti interi.

Consider an analytic function $w = f(z)$ which satisfies an algebraic differential equation

$$F(z; w, w', \dots, w^{(m)}) = 0.$$

Here F denotes a polynomial in $z, w, w', \dots, w^{(m)}$ with coefficients that may depend on any set of parameters independent of z , and the order m may be any non-negative integer.

It is clear that $w = f(z)$ satisfies not only $F = 0$, but infinitely many other algebraic differential equations as well. We shall prove in this note that $w = f(z)$ in particular satisfies an algebraic differential equation

$$G(z; w, w', \dots, w^{(M)}) = 0$$

where G is a polynomial in $z, w, w', \dots, w^{(M)}$ with *constant rational integral coefficients*, but of an order M which possibly may be greater than m . This is a rather surprising result because there exist only *countably many* distinct algebraic differential equations $G = 0$ of this kind.

The proof is purely algebraic. It applies without change to formal power series with coefficients in any field of characteristic zero. Therefore only this more general case will be considered.

1. Let L be a field of characteristic zero, and z an indeterminate. Denote by L^* the ring of all formal power series

$$f = \sum_{h=0}^{\infty} f_h z^h \quad , \quad g = \sum_{h=0}^{\infty} g_h z^h \quad , \quad \text{etc.},$$

in z with coefficients f_h, g_h , etc., in L . Here the sum and the product of such series are as usual defined by

$$f + g = \sum_{h=0}^{\infty} (f_h + g_h) z^h \quad , \quad fg = \sum_{h=0}^{\infty} \left(\sum_{k=0}^h f_k g_{h-k} \right) z^h,$$

(*) Nella seduta del 17 aprile 1971.

and the elements a of L are identified with the special power series

$$a = a + \sum_{h=1}^{\infty} 0 \cdot z^h$$

in L^* and play the role of constants.

Differentiation in L^* is defined formally by

$$\frac{d^k f}{dz^k} = f^{(k)} = \sum_{h=k}^{\infty} h(h-1)\cdots(h-k+1) f_h z^{h-k} \quad , \quad f^{(0)} = f.$$

It satisfies the usual rules for the derivatives of sums, differences, and products, and the equation

$$\frac{df}{dz} = 0$$

implies that $f = a$ is a constant, i.e. an element of L .

If

$$f = \sum_{h=0}^{\infty} f_h z^h$$

is any series in L^* , we denote by

$$K_f = Q(f_0, f_1, f_2, \dots)$$

the field which is obtained by adjoining all the coefficients f_h of f to the rational number field Q . Thus K_f is a subfield of L and depends on the particular power series f which is studied.

To shorten the text, the term *Equation* with a capital E is always to mean “*algebraic differential equation*”.

2. From now on let

$$f = \sum_{h=0}^{\infty} f_h z^h$$

be a fixed power series in L^* which satisfies an Equation

$$(F) \quad F(z; w, w', \dots, w^{(m)}) = 0$$

of arbitrary order $m \geq 0$ and with coefficients in L . On differentiating this Equation repeatedly and applying algebraic operations to the results, we can obtain infinitely many other Equations for f over L , i.e. with coefficients in L .

Whenever the polynomial F can be factorised into a product of polynomials with coefficients in L , at least one of the factors vanishes at $w = f$. It suffices therefore to consider only those Equations (F) in which F is an *irreducible* polynomial over L . We are, in addition, allowed to assume that (F) is of lowest possible order m , and that among all Equations for f over L of this order m , it is also of lowest possible degree in $w^{(m)}$, the degree n say.

The Equation for f so defined is unique up to a factor in L . For if there were two such Equations, $F = 0$ and $F^* = 0$ say, we could eliminate the terms in $w^{(m)n}$ from them and obtain a new Equation $F^{**} = 0$ for f over L which has either lower order m or lower degree n than $F = 0$.

The Equation (F) for f over L fixed by these two properties of being irreducible and having smallest m and n will be called *the defining Equation for f over L* .

To simplify the notation, we write from now on

$$F((w)) = F(z; w, w', \dots, w^{(m)})$$

and put

$$F_j((w)) = \frac{\partial}{\partial w^{(j)}} F(z; w, w', \dots, w^{(m)}) \quad (j = 0, 1, \dots, m),$$

and similarly for other differential polynomials. If (F) is the defining Equation for f over L , then evidently

$$F((f)) = 0, \quad \text{but} \quad F_m((f)) \neq 0.$$

For $F_m((w))$ does not vanish identically and has either lower order, or the same order but lower degree, than $F((w))$.

3. Let again (F) be the defining Equation for f over L . In explicit form, $F((w))$ is a finite sum

$$F((w)) = \sum_{\varrho=1}^r F_{\varrho} z^{\nu_{\varrho}} w^{\nu_{\varrho}^0} w'^{\nu_{\varrho}^1} \dots w^{(m)\nu_{\varrho}^m}$$

of monomials where the coefficients F_{ϱ} lie in L and the ν 's are non-negative integers. On substituting $w = f$, the monomials take the form

$$z^{\nu_{\varrho}} f^{\nu_{\varrho}^0} f'^{\nu_{\varrho}^1} \dots f^{(m)\nu_{\varrho}^m} = \sum_{h=0}^{\infty} \Upsilon_{\varrho h} z^h \quad (\varrho = 1, 2, \dots, r),$$

of power series in K_f^* , and the one Equation

$$F((f)) = 0$$

changes into the infinite system of homogeneous linear equations

$$\sum_{\varrho=1}^r F_{\varrho} \Upsilon_{\varrho h} = 0 \quad (h = 0, 1, 2, \dots)$$

for the coefficients F_{ϱ} of $F((w))$. Since $F((w))$ is not identically zero, these equations possess a solution F_1, \dots, F_r distinct from the trivial solution $0, \dots, 0$. On the other hand, the coefficients $\Upsilon_{\varrho h}$ of the linear equations lie in K_f . It follows then from linear algebra that there exists also a set of elements F_1^*, \dots, F_r^* of K_f distinct from $0, \dots, 0$ such that

$$\sum_{\varrho=1}^r F_{\varrho}^* \Upsilon_{\varrho h} = 0 \quad (h = 0, 1, 2, \dots).$$

Denote by F^* the new differential polynomial

$$F^*((zw)) = \sum_{q=1}^f F_q^* z^{v_q} \tau w^{v_{q0}} \tau w^{v_{q1}} \dots \tau w^{(m)} v_{qm}.$$

Then also F^* is not identically zero, and f satisfies the Equation

$$F^*((zw)) = 0$$

over K_f . The reduction process of § 2, with L replaced by K_f , may now be applied to this Equation and leads to the result that

f has a defining Equation over K_f .

From now on we shall assume that (F) itself is already this defining Equation over K_f .

4. On differentiating the formula $F((f)) = 0$ repeatedly, afterwards making use of the inequality $F_m((f)) \neq 0$ and putting $z = 0$, it can be proved that the coefficients f_h of f satisfy a recursive formula

$$(2) \quad A(h)f_h = \varphi_h(f_0, f_1, \dots, f_{h-1})$$

as soon as the suffix h is sufficiently large. Here $A(h)$ is a polynomial in h with coefficients in K_f which does not vanish identically. Further, for each suffix h , $\varphi_h(f_0, f_1, \dots, f_{h-1})$ is a polynomial in f_0, f_1, \dots, f_{h-1} , with coefficients that are linear forms in the coefficients of $F((zw))$ with rational coefficients, hence also lie in K_f . This result seems to be due to A. Hurwitz (1889). Detailed proofs can be found, e.g., in the Ph. D. thesis by Jan Popken (1935), or in my recent paper (Mahler 1971).

Denote by h_0 the smallest integer such that both

$$A(h) \neq 0 \quad \text{for } h \geq h_0,$$

and that the recursive formula (2) holds for $h \geq h_0$. This formula allows then to express successively all coefficients f_h with $h \geq h_0$ rationally with rational coefficients in

- (i) the finitely many coefficients of $F((zw))$ and $A(h)$, and
- (ii) the coefficients $f_0, f_1, \dots, f_{h_0-1}$.

Since $K_f = Q(f_0, f_1, f_2, \dots)$, we deduce then immediately the important consequence that

The field K_f is a finite extension of Q .

This extension naturally may be algebraic or transcendental.

5. From general field theory, the relation between Q and K_f can be described in the following more explicit form.

The field K_f can be obtained as a finite extension

$$K_f = Q(s_1, \dots, s_r, t)$$

of the rational number field Q . Here s_1, \dots, s_r are finitely many elements of K_f which are algebraically independent over Q , and t is a further element of K_f which is algebraic over the intermediate extension field $J = Q(s_1, \dots, s_r)$.

The integer r is the degree of transcendency of K_f over Q and is a non-negative integer. It may have the value $r = 0$, in which case K_f is an algebraic number field of finite degree over Q .

The element t of K_f can always be chosen so as to be entire over the polynomial ring $P = Q[s_1, \dots, s_r]$. The irreducible algebraic equation for t over this ring has then the form

$$(3) \quad t^d + e_1(s_1, \dots, s_r)t^{d-1} + \dots + e_d(s_1, \dots, s_r) = 0.$$

Here the degree d is some positive integer, and the coefficients

$$e_1(s_1, \dots, s_r), \dots, e_d(s_1, \dots, s_r)$$

are polynomials in the polynomial ring P . Let

$$t^{(0)} = t, t^{(1)}, \dots, t^{(d-1)}$$

be the d roots of this equation (3).

6. By hypothesis, $F((\tau w))$ is irreducible over K_f and has coefficients in K_f . Any non-zero factors of $F((\tau w))$ not involving $z, \tau w, \tau w', \dots, \tau w^{(m)}$ are irrelevant and may be omitted. Therefore, without loss of generality, we can write

$$F((\tau w)) = \Phi(z; \tau w, \tau w', \dots, \tau w^{(m)}; s_1, \dots, s_r, t) = \Phi((\tau w | s_1, \dots, s_r, t)),$$

where Φ is a polynomial in $z, \tau w, \tau w', \dots, \tau w^{(m)}, s_1, \dots, s_r$, and t , which is not identically zero and has rational coefficients.

Here we can remove t by forming the norm

$$\Psi(z; \tau w, \tau w', \dots, \tau w^{(m)}; s_1, \dots, s_r) = \Psi((\tau w | s_1, \dots, s_r)) = \prod_{\delta=0}^{d-1} \Phi((\tau w | s_1, \dots, s_r, t^{(\delta)})).$$

From the form of the equation (3) for $t^{(\delta)}$, $\Psi((\tau w | s_1, \dots, s_r))$ is then a polynomial in $z, \tau w, \tau w', \dots, \tau w^{(m)}, s_1, \dots, s_r$ with rational coefficients which does not vanish identically, and f satisfies the Equation

$$\Psi((\tau w | s_1, \dots, s_r)) = 0.$$

This is an Equation for f over the field $J = Q(s_1, \dots, s_r)$. The reduction process in § 2 enables us to derive from it also a defining Equation

$$(4) \quad X(z; \tau w, \tau w', \dots, \tau w^{(m)}; s_1, \dots, s_r) = X((\tau w | s_1, \dots, s_r)) = 0$$

for f over J . Here X is an irreducible polynomial in $z, w, w', \dots, w^{(\mu)}, s_1, \dots, s_r$ with rational coefficients, of smallest order μ and smallest degree ν say, which vanishes for $w = f$.

7. The Equation (4) may still involve the r quantities s_1, \dots, s_r in K_f which, by hypothesis, are algebraically independent over Q ; or one or more of these quantities may have disappeared in the process of forming the Equation (4). Let us assume that there exist a positive integer ρ and ρ suffixes r_1, \dots, r_ρ satisfying

$$1 \leq \rho \leq r \quad , \quad 1 \leq r_1 < r_2 < \dots < r_\rho \leq r$$

such that there exists a defining Equation for f over the field $Q(s_{r_1}, \dots, s_{r_\rho})$, but that there is no such defining Equation over any subfield $Q(s_{r_1}, \dots, s_{r_{\rho-1}})$ containing at most $\rho - 1$ of the quantities $s_{r_1}, \dots, s_{r_\rho}$.

The defining Equation for f over $Q(s_{r_1}, \dots, s_{r_\rho})$ has the form

$$Y(z; w, w', \dots, w^{(m)}; s_{r_1}, \dots, s_{r_\rho}) = Y((w | s_{r_1}, \dots, s_{r_\rho})) = 0,$$

where the differential polynomial Y , say of order m , is an irreducible polynomial in $z, w, w', \dots, w^{(m)}, s_{r_1}, \dots, s_{r_\rho}$ with rational coefficients. From the definition, Y contains the quantity s_{r_ρ} explicitly,

$$\frac{\partial Y}{\partial s_{r_\rho}} \neq 0.$$

We form the further partial derivatives

$$Y_z((w | s_{r_1}, \dots, s_{r_\rho})) = \frac{\partial}{\partial z} Y((w | s_{r_1}, \dots, s_{r_\rho})),$$

$$Y_j((w | s_{r_1}, \dots, s_{r_\rho})) = \frac{\partial}{\partial w^{(j)}} Y((w | s_{r_1}, \dots, s_{r_\rho})) \quad (j = 0, 1, \dots, m),$$

and put

$$Y^*((w | s_{r_1}, \dots, s_{r_\rho})) = Y_z((w | s_{r_1}, \dots, s_{r_\rho})) + \sum_{j=0}^m Y_j((w | s_{r_1}, \dots, s_{r_\rho})) w^{(j+1)}.$$

Then

$$Y^*((w | s_{r_1}, \dots, s_{r_\rho})) = \frac{d}{dz} Y((w | s_{r_1}, \dots, s_{r_\rho})) \quad \text{identically in } w.$$

By hypothesis, $w = f$ satisfies the Equation

$$(5) \quad Y((w | s_{r_1}, \dots, s_{r_\rho})) = 0$$

and hence also the Equation

$$(6) \quad Y^*((w | s_{r_1}, \dots, s_{r_\rho})) = 0.$$

On the other hand, since (5) is a defining equation for f and is of order m , necessarily

$$(7) \quad Y_m((f | s_{r_1}, \dots, s_{r_\rho})) \neq 0.$$

This property holds in particular in the special case when the coefficients f_h are complex numbers and f has a circle of convergence, hence when $f = f(z)$ defines an analytic function of z .

9. We had found that

$$\mathbb{K}_f = Q(f_0, f_1, f_2, \dots) = Q(s_1, \dots, s_r, t).$$

Here s_1, \dots, s_r are finitely many elements of \mathbb{K}_f which are algebraically independent over Q , and t is a root of the irreducible equation

$$(3) \quad t^d + e_1(s_1, \dots, s_r)t^{d-1} + \dots + e_d(s_1, \dots, s_r) = 0.$$

In terms of s_1, \dots, s_r , and t , each coefficient f_h of f can be written in the form

$$(9) \quad f_h = \sum_{\delta=0}^{d-1} r_{h\delta}(s_1, \dots, s_r)t^\delta, \quad = R_h(s_1, \dots, s_r, t) \quad \text{say,}$$

where the $r_{h\delta}$ are rational functions of s_1, \dots, s_r with rational coefficients.

Denote now by $\mathbf{s}_1, \dots, \mathbf{s}_r$ a set of r independent indeterminates over Q , and by \mathbf{t} the algebraic function of $\mathbf{s}_1, \dots, \mathbf{s}_r$ defined by the equation

$$(10) \quad \mathbf{t}^d + e_1(\mathbf{s}_1, \dots, \mathbf{s}_r)\mathbf{t}^{d-1} + \dots + e_d(\mathbf{s}_1, \dots, \mathbf{s}_r) = 0.$$

Further put

$$(11) \quad f_h = \sum_{\delta=0}^{d-1} r_{h\delta}(\mathbf{s}_1, \dots, \mathbf{s}_r)\mathbf{t}^\delta = R_h(\mathbf{s}_1, \dots, \mathbf{s}_r, \mathbf{t}),$$

and denote by \mathbf{f} the formal power series

$$\mathbf{f} = \sum_{h=0}^{\infty} f_h z^h.$$

This series is associated with the algebraic function field

$$\mathbb{K}_f = Q(f_0, f_1, f_2, \dots) = Q(\mathbf{s}_1, \dots, \mathbf{s}_r, \mathbf{t})$$

which, for $r=0$, becomes an algebraic number field.

By hypothesis, the Equation

$$G(z; w, w', \dots, w^{(M)}) = 0$$

for the series $w = f$ has rational coefficients which are independent of s_1, \dots, s_r , and t . It is also clear that the isomorphic mapping

$$(s_1, \dots, s_r, t) \rightarrow (\mathbf{s}_1, \dots, \mathbf{s}_r, \mathbf{t})$$

preserves all rational relations over Q . It follows therefore that not only $w = f$, but also

$$w = \mathbf{f} \text{ satisfies the Equation (G).}$$

We say that \mathbf{f} is an *indeterminate solution* of the Equation, to distinguish it from the *determinate solution* f from which we started.

10. An advantageous way of selecting the indeterminates s_1, \dots, s_r of f , and correspondingly the indeterminates $\mathbf{s}_1, \dots, \mathbf{s}_r$ of \mathbf{f} , is as follows.

Take $s_1 = f_{h_1}$ where $h_1 \geq 0$ is the smallest suffix such that f_{h_1} is transcendental over Q . Next take $s_2 = f_{h_2}$ where $h_2 > h_1$ is the smallest suffix such that f_{h_2} is transcendental over $Q(f_{h_1})$. Continuing in this manner, finally select a smallest suffix $h_r > h_{r-1}$ such that f_{h_r} is transcendental over $Q(f_{h_1}, \dots, f_{h_{r-1}})$, but that f_h for $h > h_r$ is algebraic over $Q(f_{h_1}, \dots, f_{h_r})$. This construction fixes the algebraically independent quantities

$$s_1 = f_{h_1}, \dots, s_r = f_{h_r}, \quad \text{where } 0 \leq h_1 < h_2 < \dots < h_r$$

of K_f , and t in

$$K_f = Q(f_{h_1}, \dots, f_{h_r}, t)$$

can then be chosen so as to satisfy an irreducible equation (3) with polynomial coefficients.

For the indeterminate solution \mathbf{f} we find similarly that

$$(12) \quad \mathbf{s}_1 = \mathbf{f}_{h_1}, \dots, \mathbf{s}_r = \mathbf{f}_{h_r} \quad \text{and} \quad K_{\mathbf{f}} = Q(\mathbf{f}_{h_1}, \dots, \mathbf{f}_{h_r}, t).$$

The ordered set

$$\{h\} = \{h_1, h_2, \dots, h_r\}$$

is called the *suffix set* of both f and \mathbf{f} , and r is its dimension. Both $\{h\}$ and r vary for the different solution of (G), and $\{h\}$ may have infinitely many distinct possibilities.

To give an example, the Equation

$$ww'w''' + w'^2w'' - 2ww''^2 = 0$$

has amongst others the special solutions

$$f = f_h z^h \quad (h = 0, 1, 2, \dots)$$

which belong to the suffix sets $\{0\}$, $\{1\}$, $\{2\}$, \dots , respectively.

By making use of the indeterminate solutions and choosing the indeterminates $\mathbf{s}_1, \dots, \mathbf{s}_r$ as in (12), it can be proved that

$$0 \leq r \leq M$$

for every Equation (G) of order M . This generalises the classical theorem of analysis that the general integral of an Equation of order M depends on M constants of integration.

11. Let as before

$$(G) \quad G(z; w, w', \dots, w^{(M)}) = 0$$

be the defining Equation for both f and \mathbf{f} . The derivatives

$$\left(\frac{d}{dz}\right)^h G(z; w, w', \dots, w^{(M)}), = G^{(h)}(z; w, w', \dots, w^{(M+h)}) \text{ say,}$$

are polynomials with rational coefficients in $z, w, w', \dots, w^{(M+h)}$, and f and \mathbf{f} satisfy the relations

$$G^{(h)}(z; f, f', \dots, f^{(M+h)}) = 0 \quad \text{and} \quad G^{(h)}(z; \mathbf{f}, \mathbf{f}', \dots, \mathbf{f}^{(M+h)}) = 0$$

($h = 0, 1, 2, \dots$).

Here substitute $z = 0$ so that $f^{(k)}$ and $\mathbf{f}^{(k)}$ become $k!f_k$ and $k!\mathbf{f}_k$, respectively. We obtain then the two infinite systems of algebraic equations

$$(13) \quad G^{(h)}(0; f_0, 1!f_1, 2!f_2, \dots, (M+h)!f_{M+h}) = 0 \quad (h = 0, 1, 2, \dots)$$

and

$$(14) \quad G^{(h)}(0; \mathbf{f}_0, 1!\mathbf{f}_1, 2!\mathbf{f}_2, \dots, (M+h)!\mathbf{f}_{M+h}) = 0 \quad (h = 0, 1, 2, \dots).$$

It is convenient to interpret these formulae geometrically by considering f and \mathbf{f} as points in an infinite dimensional space S , with the coordinates f_0, f_1, f_2, \dots and $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \dots$, respectively. By (13) and (14), these points lie on a manifold, \mathbf{M} say.

This manifold is essentially finite dimensional and algebraic. For by means of similar considerations as in § 4 it can be proved that there exists a positive integer k_0 , and that for every suffix $k \geq k_0$ there is a polynomial

$$\psi_k(w_0, w_1, \dots, w_{k-1})$$

with rational coefficients, such that

$$(15) \quad f_k = \psi_k(f_0, f_1, \dots, f_{k-1}) \quad \text{and} \quad \mathbf{f}_k = \psi_k(\mathbf{f}_0, \mathbf{f}_1, \dots, \mathbf{f}_{k-1}) \quad \text{for } k \geq k_0.$$

Thus the first k_0 coordinates of the points f and \mathbf{f} determine all the others rationally, entirely, and with rational coefficients. However, the integer r_0 and the polynomials ψ_k may depend on the particular solutions f and \mathbf{f} .

12. One special type of Equation is of particular interest.

Let us assume that in the defining Equation

$$(G) \quad G(w, w', \dots, w^{(M)}) = 0$$

for f and \mathbf{f} the polynomial G does not involve z explicitly and has rational integral coefficients. Also let this Equation have the exact order M so that the partial derivative

$$G_M(w, w', \dots, w^{(M)}) = \frac{\partial}{\partial w^{(M)}} G(w, w', \dots, w^{(M)})$$

does not vanish identically. Let us consider the first M coefficients $\mathbf{f}_0, \mathbf{f}_1, \dots, \mathbf{f}_{M-1}$ of \mathbf{f} as independent indeterminates. The algebraic equation

$$G(\mathbf{f}_0, 1!\mathbf{f}_1, \dots, M!\mathbf{f}_M) = 0$$

defines then f_M as an algebraic function of these indeterminates, and the higher coefficients $f_{M+1}, f_{M+2}, f_{M+3}, \dots$ become rational functions of f_0, f_1, \dots, f_M . It can in fact be proved that

$$(15) \quad f_{M+h} = \frac{H_h(f_0, f_1, \dots, f_M)}{(M+h)! G_M(f_0, 1!f_1, \dots, M!f_M)^{2h-1}} \quad (h = 1, 2, 3, \dots).$$

Here the H_h are polynomials in f_0, f_1, \dots, f_M at most of the total degree

$$c_1 h,$$

and with rational integral coefficients at most of the absolute value

$$c_2^h \cdot h!.$$

Here c_1 and c_2 are two positive constants which do not depend on h .

The solution f of (G) so defined is of maximum dimension M and belongs to the suffix set $\{0, 1, \dots, M-1\}$. On substituting for f_0, f_1, \dots, f_{M-1} any special complex or p -adic values f_0, f_1, \dots, f_{M-1} , respectively, determining f_M from

$$G(f_0, 1!f_1, \dots, M!f_M) = 0,$$

and assuming that

$$G_M(f_0, 1!f_1, \dots, M!f_M) \neq 0,$$

we obtain a determinate solution of (G). By the estimates for the degree and the height of H_h , this formal power series is in fact *convergent* if $|z|$ in the complex case and $|z|_p$ in the p -adic case is not too large.

Since (G) does not involve z explicitly, exactly analogous results hold in the neighbourhood of any other complex or p -adic point $z = c$. We may consider simultaneously these solutions for all valuations of Q . There is thus something like a global theory of the manifold M , which involves also the analytic mappings of the neighbourhoods of different points $z = 0$ and $z = c$ into each other.

REFERENCES

- A. HURWITZ (1889). - *Sur le développement des fonctions satisfaisant à une équation différentielle algébrique*, « Ann. de l'École Norm. » (3) 6, 327-332; Werke, Bd. 2, 295-298.
 K. MAHLER (1971). - *On formal power series as integrals of algebraic differential equations*, « Rend. Acc. Naz. Lincei » (8) 50, 76-89.
 J. POPKEN (1935). - *Über arithmetische Eigenschaften analytischer Funktionen*, Amsterdam, N. V. Noord-Hollandsche Uitgeversmaatschappij.