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An arithmetic remark on entire periodic functions

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For every positive number ω, there exists an odd entire transcendental function

$$f(z) = \sum_{h=0}^{\infty} a_h \frac{z^{2h+1}}{(2h+1)!}$$

with rational integral coefficients a_h such that $f(z+\omega) = f(z)$.

1.

Denote by

$$g(z) = \sum_{h=0}^{\infty} c_h \frac{z^{2h+1}}{(2h+1)!}$$

an odd entire function with real coefficients c_h where, in particular,

$$c_0 \geq 2 .$$

The odd powers of $g(z)$ allow the similar developments

$$\frac{g(z)^{2n+1}}{(2n+1)!} = \sum_{h=n}^{\infty} c_{nh} \frac{z^{2h+1}}{(2h+1)!} \quad (n = 0, 1, 2, \ldots) ,$$

and here

(1) $$c_{nn} \geq 2^{2n+1} \quad (n = 0, 1, 2, \ldots) .$$

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Next let

$$f(z) = \sum_{n=0}^{\infty} b_n \frac{g(z)^{2n+1}}{(2n+1)!}$$

where $b_0, b_1, b_2, \ldots$ denote real numbers which are determined by the following construction.

We have

$$f(z) = \sum_{n=0}^{\infty} b_n \sum_{h=0}^{\infty} c_{nh} \frac{z^{2h+1}}{(2h+1)!}$$
$$= \sum_{h=0}^{\infty} a_h \frac{z^{2h+1}}{(2h+1)!} ,$$

say, and here the new coefficients a_h are given by

$$a_h = \sum_{n=0}^{\infty} b_n c_{nh} \quad (h = 0, 1, 2, \ldots) .$$

It is thus possible to choose the coefficients b_n successively such that

$$0 < b_0 \le 2^{-1} , \quad \text{and} \quad a_0 \le 1 \quad \text{is an integer,}$$

and that for $n \ge 1$, on account of (1),

(2) $$0 \le b_n \le 2^{-(2n+1)} , \quad \text{and} \quad a_n \ne 0 \quad \text{is an integer.}$$

By this construction, $f(z)$ becomes an entire transcendental function of z. On putting

$$M(r) = \max_{|z|=r} |f(z)| , \quad M_1(r) = \max_{|z|=r} |g(z)| ,$$

by (2),

$$M(r) \le \sum_{n=0}^{\infty} 2^{-(2n+1)} \frac{M_0(r)^{2n+1}}{(2n+1)!}$$

and therefore

(3) $$M(r) < \exp\left(M_0(r)/2\right) .$$

2.

In the result so obtained, choose now

$$g(z) = \sin(2\pi z/\Omega) ,$$

where Ω is a constant satisfying

$$0 < \Omega \leq \pi .$$

Then $g(z)$ is an odd entire function with the period Ω,

$$g(z+\Omega) = g(z) ,$$

and it has a power series

$$g(z) = \sum_{h=0}^{\infty} c_h \frac{z^{2h+1}}{(2h+1)!} ,$$

where $c_0 = 2\pi/\Omega \geq 2$ as required. The preceding construction leads therefore to an odd entire transcendental function

$$f(z) = \sum_{n=0}^{\infty} b_n \frac{\left(\sin\left(\frac{2\pi z}{\Omega}\right)\right)^{2n+1}}{(2n+1)!}$$

of period Ω, and with non-vanishing integral coefficients a_h. The maximum modulus $M(r)$ of this function evidently satisfies the inequality

$$M(r) < \exp\left(\frac{e^{2\pi r/\Omega}}{2}\right) ;$$

for by the choice of $g(z)$,

$$M_1(r) < e^{2\pi r/\Omega} .$$

3.

The following result can now be proved.

THEOREM. *Let* ω *be an arbitrary positive constant. There exist two positive constants* c *and* r_0 *and an odd entire transcendental function* $f(z)$ *of period* ω,

$$f(z+\omega) = f(z) ,$$

such that the coefficients a_h *in*

$$f(z) = \sum_{h=0}^{\infty} a_h \frac{z^{2h+1}}{(2h+1)!}$$

are rational integers not zero, and that further

$$|f(z)| < e^{e^{c|z|}} \quad \textit{if} \quad |z| \geq r_0 .$$

Proof. The assertion has already been established if $0 < \omega \leq \pi$. If, however, $\omega > \pi$, then choose for k so large a positive integer that the quantity $\Omega = \omega/k$ satisfies the inequality $0 < \Omega \leq \pi$. The theorem is then valid with Ω instead of ω ; but a function of period Ω has also the period $\omega = k\Omega$.

The interest of the theorem lies in the fact that all the function values

$$f^{(\tau)}(\lambda\omega) , \quad \begin{pmatrix} \lambda = 0, 1, 2, \ldots \\ \\ \tau = 0, 1, 2, \ldots \end{pmatrix}$$

are rational integers. It is implicit in a theorem by Schneider [1, p. 49, Satz 12] that an entire transcendental function of *bounded* order and of period ω cannot have this property.

A similar proof allows to show that there exists an entire function $G(z)$ such that the function

$$F(z) = \frac{e^{G(z)}}{\Gamma(z)}$$

and all its derivatives assume rational integral values at all integral points.

Reference

[1] Theodor Schneider, *Einführung in die transzendenten Zahlen* (Grundlehren der mathematischen Wissenschaften, Band 81. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1957).

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