BULL. AUSTRAL. MATH. SOC. 10E05
VOL. 11 (1974), 121-129.

Polar analogues of two theorems by Minkowski

Kurt Mahler

Since Minkowski's time, much progress has been made in the geometry of numbers, even as far as the geometry of numbers of *convex bodies* is concerned. But, surprisingly, one rather obvious interpretation of classical theorems in this theory has so far escaped notice.

Minkowski's basic theorem establishes an *upper* estimate for the smallest positive value of a convex distance function $F(\mathrm{X})$ on the lattice of all points X with integral coordinates. By contrast, we shall establish a *lower* estimate for $F(\mathrm{X})$ at all the real points X on a suitable hyperplane

$$u_1x_1 + \ldots + u_nx_n = 1$$

with integral coefficients $u_1, \ldots, u_n$ not all zero. We arrive at this estimate by means of applying to Minkowski's Theorem the classical concept of polarity relative to the unit hypersphere

$$x_1^2 + \ldots + x_n^2 = 1 .$$

This concept of polarity allows generally to associate with known theorems on *point lattices* analogous theorems on what we call *hyperplane lattices*. These new theorems, although implicit in the old ones, seem to have some interest and perhaps further work on hyperplane lattices may lead to useful results.

In the first sections of this note a number of notations and results from the classical theory will be collected. The later

Received 9 April 1974.

sections deal then with the consequences of polarity.

1.

Let R^n be the space of all points or vectors

$$\mathbf{x} = (x_1, \ldots, x_n),\ \mathbf{y} = (y_1, \ldots, y_n),\ \mathbf{u} = (u_1, \ldots, u_n),\ \mathbf{0} = (0, \ldots, 0)\ ,$$

and so on, with real coordinates; thus $\mathbf{0}$ is the *origin*. The vector operations are as usual defined by

$$\mathbf{x} + \mathbf{y} = (x_1+y_1, \ldots, x_n+y_n),\ c\mathbf{x} = (cx_1, \ldots, cx_n),\ \mathbf{x}\cdot\mathbf{y} = x_1y_1 + \ldots + x_ny_n\ .$$

A *convex distance function* $F(\mathbf{x})$ is a function $F : R^n \to R$ with the following properties,

(1) $$F(\mathbf{0}) = 0\ ,\quad F(\mathbf{x}) > 0 \text{ if } \mathbf{x} \neq \mathbf{0}\ ;$$

(2) $$F(c\mathbf{x}) = |c|F(\mathbf{x}) \text{ for all real } c\ ;$$

(3) $$F(\mathbf{x}+\mathbf{y}) \leq F(\mathbf{x}) + F(\mathbf{y})\ .$$

The point set K in R^n defined by

$$K : F(\mathbf{x}) \leq 1$$

is then a *symmetric convex body*; that is, a bounded closed convex set in R^n which contains the origin $\mathbf{0}$ as an interior points and is symmetric in this point. Every such convex body has a *volume* $V(K)$ defined by

$$V(K) = \int \underset{K}{\cdots} \int dx_1 \ldots dx_n\ .$$

We can associate with every convex distance function $F(\mathbf{x})$ a second convex distance function $G(\mathbf{u})$ by putting

$$G(\mathbf{u}) = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{u}\cdot\mathbf{x}}{F(\mathbf{x})} = \sup_{F(\mathbf{x})=1} \mathbf{u}\cdot\mathbf{x}\ .$$

Then, conversely, also

$$F(\mathbf{x}) = \sup_{\mathbf{u}\neq\mathbf{0}} \frac{\mathbf{u}\cdot\mathbf{x}}{G(\mathbf{u})} = \sup_{G(\mathbf{u})=1} \mathbf{u}\cdot\mathbf{x}\ .$$

The set of all points

$$K^* : G(\mathbf{x}) \leq 1$$

is again a symmetric convex body and is said to be *polar reciprocal* to K with respect to the unit hypersphere

$$U : \mathbf{x}\cdot\mathbf{x} = 1 .$$

This reciprocity relation is symmetric, and K similarly is polar reciprocal to K^*.

The classical polarity relation relative to the unit hypersphere U associates with every point $\mathbf{x} = \mathbf{u}$ as *pole* the hyperplane $\mathbf{u}\cdot\mathbf{x} = 1$ as *polar*, and vice versa. One verifies easily that the pole $\mathbf{x} = \mathbf{u}$ lies in the interior, on the frontier, or on the outside of K, according as to whether the polar $\mathbf{u}\cdot\mathbf{x} = 1$ lies on the outside of K^*, is a tac-hyperplane of K^*, or penetrates into the interior of K^*. Analogous properties hold if K and K^* are interchanged.

We shall require the inequality [1, p. 108],

$$\text{(A)} \qquad n^{-n/2}\kappa_n^2 \le V(K)V(K^*) \le \kappa_n^2$$

where

$$\kappa_n = \pi^{n/2}\left(\Gamma\{(n/2)+1\}\right)^{-1}$$

denotes the volume of the unit hypersphere U. Here the upper estimate for $V(K)V(K^*)$ is best possible, but not also the lower estimate. The latter is known for $n = 2$ when it is equal to 8, but it does not seem to have yet been obtained for larger n.

2.

Denote by Λ the point lattice of all points $\mathbf{x}$ in R^n with integral coordinates $x_1, \ldots, x_n$. One main problem of the geometry of numbers deals with the question whether a given set Σ of points intersects Λ in a point $\mathbf{x} \ne \mathbf{0}$.

For the case that Σ is a symmetric convex body $K : F(\mathbf{x}) \le 1$, Minkowski himself already obtained two very basic results which have allowed important applications to the theory of algebraic number fields and to diophantine approximations.

Minkowski's first theorem states that [1, pp. 33-34],

(I) *If* $V(K) \geq 2^n$ *, then* K *contains at least one point* $\mathrm{x} \neq \mathbf{0}$ *of* Λ .

This theorem is contained in the very deep second theorem of Minkowski on the successive minima of K in Λ . Here the successive minima are obtained by the following construction.

Select a lattice point $\mathrm{x}^{(1)} \neq \mathbf{0}$ for which

$$F(\mathrm{x}^{(1)}) = m_1$$

is a minimum. Next choose a lattice point $\mathrm{x}^{(2)}$ which is linearly independent of $\mathrm{x}^{(1)}$ and for which, under this restriction,

$$F(\mathrm{x}^{(2)}) = m_2$$

is as small as possible. Generally, for $k = 2, 3, \ldots, n$, if the lattice points $\mathrm{x}^{(1)}, \ldots, \mathrm{x}^{(k-1)}$ and the corresponding minima $m_1, \ldots, m_{k-1}$ have already been defined, let $\mathrm{x}^{(k)}$ be a lattice point which is linearly independent of $\mathrm{x}^{(1)}, \ldots, \mathrm{x}^{(k)}$ such that

$$F(\mathrm{x}^{(k)}) = m_k$$

is as small as possible.

The n minima $m_1, \ldots, m_n$ so defined are called the *successive minima* of K on Λ , and they are uniquely determined. From the definition,

$$0 < m_1 \leq m_2 \leq \ldots \leq m_n .$$

Minkowski's second theorem asserts now that [*1*, p. 52],

(II)
$$\frac{2^n}{n!} \leq m_1 m_2 \ldots m_n V(K) \leq 2^n ,$$

where the right-hand inequality is best possible.

The n lattice points $\mathrm{x}^{(1)}, \ldots, \mathrm{x}^{(n)}$, at which the successive minima $m_1, \ldots, m_n$ are attained, will not in general form a basis of Λ . For such bases the following slightly weaker result holds [*1*, p. 61].

There exists a basis $x^{[1]}, \ldots, x^{[n]}$ *of* Λ *for which*

$$\text{(III)} \qquad \frac{2^n}{n!} \le F\left(x^{[1]}\right)F\left(x^{[2]}\right) \ldots F\left(x^{[n]}\right)V(K) \le 2n! \; .$$

Here neither of the two bounds is in general best possible.

3.

The three theorems (I), (II), and (III) can naturally also be applied to the body K^* which is polar reciprocal to K. The polarity relation relative to the unit hypersphere U leads then immediately to new properties of the original convex body K.

Denote by V the set of all hyperplanes

$$u \cdot x = 1$$

where u lies in Λ, thus is a point with integral coordinates. As we noted already, with the point u as pole, this hyperplane is the polar relative to U. It is convenient to exclude the improper hyperplane $0 \cdot x = 1$ which corresponds to the origin 0.

Assume, firstly, that the volume of the polar reciprocal body K^* satisfies the inequality

$$\text{(4)} \qquad V(K^*) \ge 2^n \; .$$

Then, by (I), K^* contains a point $u \ne 0$ of the point lattice Λ; thus u lies either in the interior or on the frontier of K^*. By what was said in §1, this means that the hyperplane $u \cdot x = 1$ does not meet K in any interior point. Hence at all points of this hyperplane $F(x)$ is at least 1. Since, by (A), the inequality (4) is certainly satisfied if

$$V(K) \le (4n)^{-n/2}\kappa_n^2 \; ,$$

the following result holds.

THEOREM 1. *Let* $K : F(x) \le 1$ *be a convex body of volume*

$$\text{(5)} \qquad V(K) \le (4n)^{-n/2}\kappa_n^2 \; .$$

Then there exists an integral vector $u \ne 0$ *such that*

$F(x) \ge 1$ *at all real points* x *satisfying* $u \cdot x = 1$.

4.

To give an example to Theorem 1, let $n = m + 1$; let $a_1, \ldots, a_m$, and t be real numbers where $t > 1$; and let $F(\mathrm{X})$ be the convex distance function

$$F(\mathrm{X}) = \max\Big\{t^{-1}|x_1|, \ldots, t^{-1}|x_m|, \\ t^m\{16(m+1)\}^{(m+1)/2}\kappa_{m+1}^{-2}|a_1x_1 + \ldots + a_mx_m+x_{m+1}|\Big\} .$$

The convex body $K : F(\mathrm{X}) \le 1$ evidently has the volume

$$(2t)^m \cdot 2t^{-m}\{16(m+1)\}^{-(m+1)/2}\kappa_{m+1}^2 = \{4(m+1)\}^{-(m+1)/2}\kappa_{m+1}^2 ,$$

so that the condition (5) of Theorem 1 is satisfied.

Hence, by this theorem, there exist integers $u_1, \ldots, u_m, u_{m+1}$ not all zero such that $F(\mathrm{X}) \ge 1$ at all real points $\mathrm{X} = (x_1, \ldots, x_m, x_{m+1})$ satisfying

(6) $$u_1x_1 + \ldots + u_mx_m + u_{m+1}x_{m+1} = 1 .$$

Hence, whenever X lies on this hyperplane while at the same time

(7) $$|x_1| < t, \ldots, |x_m| < t ,$$

then necessarily

(8) $$|a_1x_1 + \ldots + a_mx_m+x_{m+1}| \ge t^{-m}\{16(m+1)\}^{-(m+1)/2}\kappa_{m+1}^2 .$$

Here we can immediately assert that

$$u_{m+1} \neq 0 .$$

For assume that $u_{m+1} = 0$; then at least one of the integers $u_1, \ldots, u_m$ is distinct from zero. Since by hypothesis $t > 1$, there exist real numbers $x_1, \ldots, x_m$ which satisfy both the inequalities (7) and the equation

$$u_1x_1 + \ldots + u_mx_m = 1 .$$

On defining now x_{m+1} by the formula

$$a_1x_1 + \ldots + a_mx_m + x_{m+1} = 0 \ ,$$

we obtain a contradiction to the inequality (8).

Since then $u_{m+1} \neq 0$, the equation (6) shows that

$$x_{m+1} = u_{m+1}^{-1}\left(1-u_1x_1 - \ldots - u_mx_m\right) \ ,$$

so that we arrive at the following rather strange result.

Let $a_1, \ldots, a_m$ *, and* t *, be real numbers where* $t > 1$ *. Then there exist integers* $u_1, \ldots, u_m, u_{m+1}$ *, where* $u_{m+1} \neq 0$ *, such that*

$$\left|\left(a_1u_{m+1}-u_1\right)x_1 + \ldots + \left(a_mu_{m+1}-u_m\right)x_m+1\right| \geq t^{-m}\{16(m+1)\}^{-(m+1)/2}\kappa_{m+1}^2\left|u_{m+1}\right|$$

for every set of m *real numbers* $x_1, \ldots, x_m$ *satisfying*

$$\left|x_1\right| < t, \ \ldots, \ \left|x_m\right| < t \ .$$

5.

In a similar manner as we deduced Theorem 1 from Minkowski's Theorem (I), we shall now establish polar analogues of the Theorems (II) and (III). For this purpose we first apply these theorems to the symmetric convex body $K^* : G(\mathbf{x}) \leq 1$ which is polar reciprocal to $K : F(\mathbf{x}) \leq 1$.

To begin with the Theorem (II), let $m_1^*, \ldots, m_n^*$ be the successive minima of K^* in Λ , and let $\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(n)}$ be a set of n linearly independent points in Λ at which the minima are attained,

$$m_k^* = G\left(\mathbf{u}^{(k)}\right) \quad (k = 1, 2, \ldots, n) \ .$$

Thus the points $m_k^{*-1}\mathbf{u}^{(k)}$ satisfy the equations

$$G\left(m_k^{*-1}\mathbf{u}^{(k)}\right) = 1 \quad (k = 1, 2, \ldots, n) \ ,$$

hence lie on the frontier $G(\mathbf{x}) = 1$ of K^* . The polar relative to the unit hypersphere U of the pole $m_k^{*-1}\mathbf{u}^{(k)}$ is the hyperplane

$$m_k^{*-1}\mathbf{u}^{(k)}\cdot\mathbf{x} = 1 ,$$

and by §1, this hyperplane is a tac-hyperplane of K. It follows that the parallel hyperplane

$$\mathbf{u}^{(k)}\cdot\mathbf{x} = 1$$

is a tak-hyperplane of the convex body defined by the inequality

$$F(\mathbf{x}) \le m_k^{*-1} ,$$

and that therefore

$$m_k^{*-1} = \inf_{\mathbf{u}^{(k)}\cdot\mathbf{x}=1} F(\mathbf{x}) \quad (k = 1, 2, \ldots, n) .$$

Now, by Minkowski's Theorem (II) applied to K^*,

$$\frac{2^n}{n!} \le m_1^* m_2^* \ldots m_n^* V(K^*) \le 2^n ,$$

and here, by (A),

$$n^{-n/2}\kappa_n^2 \le V(K)V(K^*) \le \kappa_n^2 .$$

Hence

$$(4n)^{-n}\kappa_n^2 \le \left(m_1^* m_2^* \ldots m_n^*\right)^{-1} V(K) \le 2^{-n} n!\kappa_n^2 .$$

This result may be formulated as follows.

THEOREM 2. *Let* $K : F(\mathbf{x}) \le 1$ *be a symmetric convex body. Then there exist* n *linearly independent lattice points* $\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(n)}$ *such that*

$$(4n)^{-n}\kappa_n^2 \le V(K) \prod_{k=1}^{n} \left\{ \inf_{\mathbf{u}^{(k)}\cdot\mathbf{x}=1} F(\mathbf{x}) \right\} \le 2^{-n} n!\kappa_n^2 .$$

On applying the same considerations to the inequality (III) instead of (II), we obtain the following similar result.

THEOREM 3. *Let* $K : F(\mathbf{x}) \le 1$ *be a symmetric convex body. Then there exists a basis* $\mathbf{u}^{[1]}, \ldots, \mathbf{u}^{[n]}$ *of the lattice* Λ *such that*

$$(2n!)^{-1}n^{-n/2}\kappa_n^2 \le V(K)\prod_{k=1}^{n}\left(\inf_{\mathbf{u}^{[k]}\cdot\mathbf{x}=1} F(\mathbf{x})\right) \le 2^{-n}n!\kappa_n^2 .$$

In both Theorems 2 and 3 the coordinates of $\mathbf{x}$ may run over arbitrary real numbers.

Reference

[1] C.G. Lekkerkerker, *Geometry of numbers* (Bibliotheca Mathematica, . Wolters-Noordhoff, Groningen; North-Holland, Amsterdam, London; 1969).

Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra, ACT.