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On Some Special Decimal Fractions

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For a special countable set of irrational numbers, infinitely many integral multiples are constructed the decimal expansions of which begin with very large numbers of the digit 9.

Real irrational numbers have the well-known property that the fractional parts of their multiples lie dense, and even are uniformly distributed, between 0 and 1. In a different direction I recently proved [2] the following result: To every positive integer n there exists a second positive integer $P = P(n)$ such that, if α is any real irrational number, then there is a positive integer $p = p(\alpha, n)$ satisfying $1 \le p \le P(n)$ such that every possible sequence of n digits $0, 1, 2, \ldots, 9$ occurs infinitely often in the decimal expansion of $p\alpha$.

In the present note, I shall establish a result of a somewhat different kind. Let $f(x)$ be a positive integral valued polynomial of degree $m \ge 1$, and let $\sigma(f)$ be the decimal fraction obtained by writing the decimal forms of $f(1)$, $f(2), f(3), \ldots$ successively after the decimal point. *Then, for every sufficiently large positive integer N, among the first $10^{N/m}$ digits after the decimal point of*

$$((10^1 - 1)(10^2 - 1) \cdots (10^N - 1))^{m+1}\sigma,$$

there are at most $(m + 1)N^3$ digits distinct from 9.

Almost forty years ago I had studied these numbers [1] and proved that they are transcendental, but are not Liouville numbers. In the present note I apply both notations and results of this old paper.

1.

We begin with an almost trivial lemma.

Lemma *Let*

$$\sigma = r_0 - \sum_{\nu=1}^{\infty} r_\nu \cdot 10^{-i_\nu}$$

be a convergent series where the r_ν are positive rational numbers, and the i_ν are strictly increasing positive integers. Denote by d_N a common denominator of r_0, $r_1, \ldots, r_N$, and by e_N and ε_N positive integers such that

$$d_N \cdot \max(r_0, r_1, \ldots, r_N) < 10^{e_N}, \qquad d_N \cdot \sum_{\nu=N+1}^{\infty} r_\nu 10^{-i_\nu} < 10^{-\varepsilon_N}.$$

If

$$\varepsilon_N > Ne_N,$$

then at least $\varepsilon_N - Ne_N$ of the first ε_N digits after the decimal point in the decimal expansion of $d_N\sigma$ are equal to 9.

Proof The positive integer $d_N r_0$ can be written as

$$d_N r_0 = (d_N r_0 - 1) + 0.999\ldots,$$

where the decimal fraction has only digits 9. Hence

$$d_N\sigma = (d_N r_0 - 1) + 0.999\ldots - \sum_{\nu=1}^{N} d_N r_\nu \cdot 10^{-i_\nu} - d_N \cdot \sum_{\nu=N+1}^{\infty} r_\nu \cdot 10^{i_\nu}.$$

Here, by the definition of e_ν and ε_ν, each of the decimal representations for the integers $d_N r_1, d_N r_2, \ldots, d_N r_N$ contains at most e_N digits distinct from 0, and the decimal expansion of the convergent series

$$d_N \cdot \sum_{\nu=N+1}^{\infty} r_\nu \cdot 10^{-i_\nu}$$

has only zero digits in the first ε_N places after the decimal point. On account of the term 0.999 ... the assertion follows immediately.

2.

The numbers $\sigma(f)$ of my paper [1] were defined as follows. Denote by $f(x)$ a polynomial of the exact degree $m \geq 1$ which for positive integral x

assumes only nonnegative integral values and write

$$f_h(x) = \Delta^h f(x) = \sum_{H=0}^{h} (-1)^H \binom{h}{H} f(x+h-H) \qquad (h = 0, 1, 2, \ldots)$$

for the successive differences of $f(x)$. We need consider these differences only for $0 \leq h \leq m$ because those with $h \geq m+1$ vanish identically. All these differences have integral values for all positive integers x, and they are moreover positive if x is sufficiently large.

Without loss of generality, we impose the stronger restriction that $f(x)$ is strictly increasing and positive for $x \geq 1$ and that moreover

$$f_h(x) > 0 \qquad \text{for} \quad 0 \leq h \leq m \text{ and } x \geq 1.$$

It follows in particular that, for $y \geq f(1) \geq 1$, there exists the inverse function $x = g(y)$ of $y = f(x)$, and here also $g(y)$ is strictly increasing.

A little more can be said. We can write $f(x)$ in the explicit form

$$f(x) = \alpha^{-m} x^m (1 + \alpha_1 x^{-1} + \alpha_2 x^{-2} + \cdots + \alpha_m x^{-m}),$$

where α is a certain positive number and $\alpha_1, \alpha_2, \ldots, \alpha_m$ are certain real constants. Hence it follows that for sufficiently large y,

$$g(y) = \alpha y^{1/m} + O(1). \tag{1}$$

3.

For every positive integer k the function value $f(k)$ can be written as a finite decimal

$$f(k) = \sum_{\lambda=0}^{M_k} d_{k\lambda} 10^{M_k - \lambda} = d_{k0} d_{k1} \cdots d_{kM_k},$$

where the coefficients $d_{k\lambda}$ are decimal digits 0, 1, ..., 9, where in particular

$$d_{k0} > 0 \qquad \text{for all } k,$$

and where the numbers M_k are nondecreasing nonnegative integers.

We associate now with the polynomial $f(x)$ the infinite decimal fraction

$$\sigma(f) = 0 \cdot d_{10} d_{11} \cdots d_{1M_1} d_{20} d_{21} \cdots d_{2M_2} d_{30} d_{31} \cdots d_{M_3} \cdots.$$

By way of example, the polynomial

$$\phi(x) = x(x+1)/2$$

has the required properties, and with it is associated the decimal fraction

$$\sigma(\phi) = 0.1\ 3\ 6\ 10\ 15\ 21\ 28\ 36\ 45\ 55\ 66\ 78\ 91\ 105\ 120\ 136 \ldots.$$

From my old paper [1] I take a strongly convergent series for $\sigma(f)$. Denote by n the positive integer for which

$$10^{n-1} \leq f(1) \leq 10^n - 1,$$

and put

$$j_{n-1} = 0 \quad \text{and} \quad j_\nu = [g(10^\nu - 1)] \qquad \text{for} \quad \nu = n, n+1, n+2, \ldots;$$

further write

$$J_\nu = \sum_{\mu=n}^{\nu-1} \mu(j_\mu - j_{\mu-1}) = (\nu - 1)j_{\nu-1} - \sum_{\mu=n}^{\nu-2} j_u$$

$$\text{for} \quad \nu = n+1, n+2, n+3, \ldots .$$

With this notation

$$\sigma(f) = \sum_{k=1}^{j_n} f(k)10^{-kn} + \sum_{\nu=n+1}^{\infty} 10^{-J_\nu + j_{\nu-1}} \sum_{k=j_{\nu-1}+1}^{j_\nu} f(k)10^{-k\nu}.$$

Here the finite sums can be summed by means of a formula from difference calculus, giving the formula

$$\sigma(f) = \sum_{h=0}^{m} f_h(1)(10^n - 1)^{-(h+1)}$$

$$- \sum_{\nu=n+1}^{\infty} 10^{-J_\nu} \sum_{h=0}^{m} f_h(j_{\nu-1} + 1)((10^{\nu-1} - 1)^{-(h+1)} - (10^\nu - 1)^{-(h+1)}). \tag{2}$$

4.

On putting

$$r_0 = \sum_{h=0}^{m} f_h(1)(10^n - 1)^{-(h+1)},$$

$$r_{\nu-n} = \sum_{h=0}^{m} f_h(j_{\nu-1} + 1)((10^{\nu-1} - 1)^{-(h+1)} - (10^\nu - 1)^{-(h+1)})$$

$$\text{for} \quad \nu \geq n+1,$$

$$i_\nu = J_{n+\nu} \qquad \text{for} \quad \nu \geq 1,$$

the formula (2) can be written as

$$\sigma(f) = r_0 - \sum_{\nu=1}^{\infty} r_\nu 10^{-i_\nu}, \tag{3}$$

a series of the same form as in the lemma. From their definitions, all the numbers $r_0, r_1, r_2, \ldots$ are positive and rational. The coefficients $f_h(1)$ and $f_h(j_{\nu-1}+1)$ are positive integers. If further d_N denotes the product

$$d_N = ((10^1 - 1)(10^2 - 1) \cdots (10^N - 1))^{m+1}, \tag{4}$$

then d_N is a common denominator of the $N+1$ rational numbers $r_0, r_1, \ldots, r_N$, and

$$d_N < (10^{1+2+\cdots+N})^{m+1} = 10^{(m+1)N(N+1)/2}.$$

In order to make use of the lemma, we require upper estimates for the numbers e_ν and ε_ν. Such estimates can be derived from the formula (1) for $g(y)$. It implies that for large ν

$$j_\nu = \alpha \cdot 10^{\nu/m} + O(1) \qquad \text{and} \qquad J_\nu = \alpha(\nu - 1) \cdot 10^{(\nu-1)/m} + O(10^{(\nu-1)/m}), \tag{5}$$

This implies that there are two positive constants c and C such that for large ν,

$$c\nu \cdot 10^{\nu/m} \leq i_\nu \leq C\nu \cdot 10^{\nu/m}. \tag{6}$$

Further, for $h = 0, 1, \ldots, m$, and for large x,

$$f_h(x) = O(x^m),$$

hence, by (5), for all such values of h and for large ν,

$$f_h(j_{\nu-1} + 1) = O(10^\nu).$$

On the other hand, for such h and ν,

$$(10^{\nu-1} - 1)^{-(h+1)} - (10^\nu - 1)^{-(h+1)} = O(10^{-\nu}).$$

Therefore all the sums $r_0, r_1, r_2, \ldots$ are bounded positive numbers, say not larger than $10^p - 1$ where p is some positive integer. This means that for sufficiently large integers N the products $d_N r_0, d_N r_1, d_N r_2, \ldots$ are positive integers not greater than

$$10^{(m+1)N2} - 1,$$

therefore by (6) that

$$0 < \sum_{\nu=N+1}^{\infty} d_N r_\nu \cdot 10^{-i_\nu} < 10^{-10^{N/m}}.$$

The hypothesis of the lemma is thus satisfied with

$$e_N = (m+1)N^2 \qquad \text{and} \qquad \varepsilon_N = 10^{N/m}.$$

Here $\varepsilon_N > Ne_N$ for all sufficiently large N, and hence the lemma leads to the following result.

Theorem *The decimal fraction $\sigma(f)$ belonging to a polynomial $f(x)$ of degree m has the following property. To every sufficiently large positive integer N there exists a positive integer d_N of at most $(m+1)N(N+1)/2$ decimal places such that at least $10^{N/m} - (m+1)N^3$ of the first $10^{N/m}$ digits after the decimal point in the decimal expansion of $d_N\sigma(f)$ are equal to 9.*

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[1] Kurt Mahler, Arithmetische Eigenschaften einer Klasse von Dezimalbrüchen, *Proc. Akad. Wetenschappen, Amsterdam* **40** (1937), 421–428.

[2] Kurt Mahler, Arithmetical properties of the digits of the multiples of an irrational number, *Bull. Austral. Math. Soc.* **8** (1973), 191–203.