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Fifty Years as a Mathematician*

K. MAHLER

Australian National University, Canberra ACT, 2600, Australia

During the height of the great German Inflation, I began my university studies in the autumn of 1923 at the University of Frankfurt. I remained there for three semesters, and then, in the spring of 1925, went to the University of Göttingen.

Although there was only a small number of students in the Mathematics Department at Frankfurt, it had outstanding teachers, the Professors Dehn, Hellinger, Epstein, Szász, and Siegel. From Siegel I learned most, particularly in analytic number theory and related parts of function theory. He introduced me to Diophantine equations and approximations, and, in his great paper of 1929, to transcendental numbers.

Göttingen, during my stay from 1925 to 1933, had a much larger mathematical department, and it was at that time a centre of world mathematics, with many distinguished visitors from abroad. Of particular relevance for my later research was what I learned from Courant about direct methods in the calculus of variations, and from Emmy Noether about modern algebra and in particular about fields with valuations and p-adic numbers.

After the coming of Hitler in 1933, I left Göttingen where, since my Frankfurt doctorate in 1927, I had been doing research, chiefly on transcendental numbers and Diophantine approximations.

On the invitation of Mordell, I spent the session of 1933–1934 at the University of Manchester. Then, on the invitation of van der Corput, I went for the next two years 1934 to 1936 to the University of Groningen in the Netherlands. After another year's leave of absence, due to illness, I returned to the University of Manchester in the autumn of 1937 and was to stay there until 1963 when I went for the next five years to the Australian National University at Canberra until my retirement at 65. Between 1968 and 1972 I held a professorship at the Ohio State University at Columbus, Ohio. Finally I returned in 1972 for retirement to the Australian National University in Canberra.

* I prepared these notes about 1973 and have left them unchanged, except for some corrections. However, the list of my publications has been extended to 1981.

In these notes I shall report on my research on three subjects where great progress was made in this century.

I. Rational Approximations of Algebraic Numbers.

(1) Let ζ be a real irrational number, and let p and $q > 0$ be integers. Denote by $t(\zeta)$ the least upper bound of all positive numbers τ for which

$$\left|\zeta - \frac{p}{q}\right| \leqslant q^{-\tau}$$

has infinitely many solutions p/q. From the continued fraction for ζ,

$$t(\zeta) \geqslant 2,$$

and $t(\zeta)$ may in fact for suitable ζ be any number in the closed interval $[2, \infty]$.

Next let ζ be a real *algebraic* number of degree $n \geqslant 2$. Denote by

$$A(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \qquad \text{where } a_0 a_n \neq 0,$$

an irreducible polynomial with integral coefficients of which ζ is a zero. It cannot be a multiple zero. Hence

$$A(z) = (z - \zeta)\, B(z),$$

where $B(z)$ is a polynomial of degree $n-1$ with real coefficients such that

$$B(\zeta) \neq 0.$$

Hence three positive constants $c_1, c_2,$ and c_3 exist such that

$$c_1 \leqslant |B(z)| \leqslant c_2 \qquad \text{if} \quad |z - \zeta| \leqslant c_3.$$

Assume in particular that p and $q > 0$ are integers satisfying

$$\left|\zeta - \frac{p}{q}\right| \leqslant c_3.$$

$A(z)$ has no rational zero. Therefore the rational number $A(p/q)$ has the form Nq^{-n}, where the integer N is not zero and so $|N| \geqslant 1$. It follows that

$$\left|\zeta - \frac{p}{q}\right| = \left|A\left(\frac{p}{q}\right)\Big/B\left(\frac{p}{q}\right)\right| \geqslant c_2^{-1} q^{-n}.$$

This estimate has been proved for $|\zeta - p/q| \leqslant c_3$, but it is clear that if c denotes the larger one of c_2 and c_3^{-1}, then

$$\left|\zeta - \frac{p}{q}\right| \geqslant (cq^n)^{-1}.$$

This inequality was obtained by J. Liouville in 1844 who used it to construct the first examples of transcendental numbers. It implies that

$$t(\zeta) \leqslant n$$

for all real algebraic numbers ζ of degree n.

(**2**) Associated with the polynomial $A(z)$ is the binary form

$$A(x, y) = A\left(\frac{x}{y}\right)y^n = a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n$$

which likewise is irreducible. Assume that for some integer $m \neq 0$ the Diophantine equation

$$A(p, q) = a_0p^n + a_1p^{n-1}q + \cdots + a_nq^n = m \tag{1}$$

has infinitely many solutions in integers p, q. Since the case $q = 0$ is trivial and since we may, if necessary, replace p, q, m by $-p$, $-q$, $(-1)^n m$ we are allowed to assume that q is positive and arbitrarily large. Now

$$A\left(\frac{p}{q}\right) = mq^{-n}$$

is arbitrarily small. Hence the quotient p/q must finally be arbitrarily close to one of the zeros of $A(z)$, say, to the zero ζ, and hence it satisfies the inequality

$$\left|\zeta - \frac{p}{q}\right| \leqslant c_3.$$

It follows then from the earlier estimates that the inequality

$$\left|\zeta - \frac{p}{q}\right| = \left|A\left(\frac{p}{q}\right)\middle/B\left(\frac{p}{q}\right)\right| \leqslant |m|(c_1q^n)^{-1}$$

has infinitely many solutions in integers $p, q > 0$. This means that

$$t(\zeta) \geqslant n,$$

hence by Liouville's formula that

$$t(\zeta) = n.$$

This equation represents thus a necessary condition for (1) to have infinitely many integral solutions p, q. It is in fact satisfied for indefinite *quadratic* forms $A(x, y)$, as the example of Pell's equation shows.

(**3**) It was the great achievement of A. Thue in 1908 to prove that (1) has for every $m \neq 0$ at most finitely many integral solutions p, q if the degree n is at least 3. This he proved by showing that for $n \geqslant 3$ and every zero ζ of $A(z)$

$$t(\zeta) \leqslant \frac{n}{2} + 1 < n.$$

After Thue this estimate was successively improved by C. L. Siegel in 1921, by F. J. Dyson and A. O. Gelfond in 1947, and by F. K. Roth in 1955 who proved that

$$t(\zeta) \leqslant 2\sqrt{n}, \qquad t(\zeta) \leqslant \sqrt{2n}, \qquad t(\zeta) = 2,$$

respectively. Here Roth's formula $t(\zeta) = 2$ is of course best possible.

The proofs of all these results are similar, but they become progressively more and more complicated. It will suffice to sketch the idea of Thue's proof.

For this purpose denote by

$$\tau > \frac{n}{2} + 1$$

a constant and assume that the inequality

$$\left|\zeta - \frac{p}{q}\right| < q^{-\tau}$$

has infinitely many solutions p/q, where $q > 0$. Let p_1/q_1 and p_2/q_2 be two such solutions with large denominators $q_1 > 0$ and $q_2 > 0$. By means of Dirichlet's principle (the "Schubfachprinzip") one constructs polynomials of the form

$$R(x, y) = \sum_{h=0}^{m+r} \sum_{k=0}^{1} R_{hk} x^h y^k \not\equiv 0$$

of high degree $m + r$ in x and of degree 1 in y with the following properties.

(a) *The coefficients R_{hk} are integers with not too large absolute values.*

(b) *Identically in x and y,*

$$R(x, y) = (x - \zeta)^r F(x, y) + (y - \zeta)\, G(x, y), \tag{2}$$

where F and G are polynomials at most of degree $m + r$ in x and 1 in y which have coefficients in the algebraic number field $\mathbb{Q}(\zeta)$ the coefficients of which have not too large absolute values.

(c) *The rational number*

$$R_0 = R(p_1/q_1, p_2/q_2)$$

does not vanish.

Now substitute in (2),

$$x = p_1/q_1 \qquad \text{and} \qquad y = p_2/q_2.$$

By (c), the left-hand side is a rational number distinct from 0 with the denominator $q_1^{m+r} q_2$; hence

$$|R_0| \geqslant (q_1^{m+r} q_2)^{-1}.$$

On the other hand, up to a factor which is not too large, the right-hand side of (2) has an absolute value not greater than

$$|p_1/q_1 - \zeta|^r + |p_2/q_2 - \zeta| < q_1^{-r\tau} + q_2^{-\tau}.$$

If now q_1 and q_2 are sufficiently large, the two integers m and r can be chosen such that

$$(q_1^{m+r} q_2)^{-1} > q_1^{-r\tau} + q_2^{-\tau}.$$

This contradiction proves that our assumption was false, so proving Thue's theorem.

The main difficulty of this proof does not lie in the construction of the polynomial $R(x, y)$, but in establishing property (c). It is solved by showing that, even if the originally constructed polynomial $R(x, y)$ has not yet property (c), it is possible to find a not too large integer $d \geqslant 0$ such that the derivative $(\partial/\partial x)^d R(x, y)$ has the three properties (a), (b), and (c).

In the proofs of the estimates by Siegel, Dyson, and Gelfond, the polynomial $R(x, y)$ need no longer be of degree 1 in y, but may be of higher degree. In the much more difficult proof of Roth's theorem, $R(x, y)$ is replaced by a polynomial in an arbitrarily large number of variables, and one substitutes for these variables an equal number of rational approximations of ζ. The essential difficulty lies again in the proof of an analogue to property (c).

(**4**) In his paper of 1921, Siegel already generalised the problem to that of the approximation of an arbitrary real or complex algebraic number by the elements of a fixed algebraic number field of finite degree over $\mathbb{Q}$. This work he further extended in his fundamental paper of 1929 where he proved the following theorem.

Let $F(x, y)$ be a polynomial with algebraic coefficients such that the algebraic curve

$$C: \qquad F(x, y) = 0$$

has positive genus. Let further K be any algebraic number field of finite degree over the rational field. Then there are at most finitely many points (x, y) on C such that x and y are integers in K. The examples of linear Diophantine equations and of Pell's equation show that this theorem does not apply to all curves of genus 0, but the exceptional cases can be fully characterised.

(**5**) I come now to a short discussion of my own work on the approximation of algebraic numbers and of Diophantine equations.

During my studies at the University of Frankfurt I had learned from Siegel the results by Thue and himself, and at the University of Göttingen I learned from Emmy Noether about valuation theory and in particular about p-adic numbers. I knew Ostrowski's theorem that the rational field $\mathbb{Q}$ has essentially only the absolute valuation $|\alpha|$ and for every prime p the p-adic valuation $|\alpha|_p$. The completion of $\mathbb{Q}$ relative to $|\alpha|$ is the real field $\mathbb{R}$, while that relative to $|\alpha|_p$ is Hensel's field of p-adic numbers. Different primes generate different p-adic fields, all, like $\mathbb{R}$, being extensions of the rational field $\mathbb{Q}$. The interrelation of all these valuations is, for every element $a \neq 0$ of $\mathbb{Q}$, expressed by the basic product formula

$$|a| \prod_p |a|_p = 1 \tag{3}$$

where the product extends over all different primes.

K. Hensel introduced the p-adic numbers in the 1890s and applied them in particular to the theory of algebraic number fields. During the following decades they found applications in more and more branches of mathematics.

In the 1920s Hasse gave a lecture on their importance in algebra and number theory. The only field in which, at that time, he could not forsee any uses was that of Diophantine approximations.

As so often, such predictions may be shown to be false, and I gave this proof.

During a rained-out Whitsun vacation in 1929 on a North Sea Island when it was unpleasant out of doors and I had nothing else to do, I tried to establish a p-adic analogue to Siegel's theorem of 1921 on the rational approximations of algebraic numbers. Following his method, except for certain changes which allowed to avoid the use of real numbers, I could prove the following special result.

Let $A(z)$ and $A(x, y)$ have the same meaning as before, and let further P be a prime and ζ_P be a P-adic root of $A(z) = 0$. Then the inequality

$$\left|\zeta_P - \frac{p}{q}\right|_P < \max(|p|, |q|)^{-2\sqrt{n}}$$

has at most finitely many solutions (p, q) in rational integers p and q.

From this estimate it follows that, when p and q are not both divisible by P, $A(p, q)$ cannot be divisible by a higher power of P than

$$C.\max(|p|, |q|)^{2\sqrt{n}},$$

where $C > 0$ is a constant independent of p and q.

During the following two years I could step by step generalise these results by considering not one, but any finite number of distinct valuations of $\mathbb{Q}$. In 1931 I finally arrived at the following theorem.

Let $P_1, P_2, \ldots, P_t$ be finitely many distinct primes, and let the integers p and q be relatively prime. Denote by $Q(p, q)$ the largest product

$$P_1^{a_1} P_2^{a_2} \cdots P_t^{a_t}$$

with non-negative integral exponents $a_1, a_2, \ldots, a_t$ which divides $A(p, q)$. Then

$$\frac{|A(p, q)|}{Q(p, q)} > c. \max(|p|, |q|)^{n-2\sqrt{n}}$$

where $c > 0$ is a constant which does not depend on p and q. Here the exponent $n - 2\sqrt{n}$ may for $n = 3$ be replaced by any number less than $\frac{1}{2}$ and for $n = 4$ by any number less than 1.

This theorem remains valid even when $A(x, y)$ is reducible, provided it has at least three linear factors no two of which differ only by a constant factor. Under the same hypothesis the greatest prime factor of $A(p, q)$ tends to infinity if both

$$(p, q) = 1 \qquad \text{and} \qquad \max(|p|, |q|) \to \infty.$$

It also follows that the number of solutions of

$$A(p, q) = m, \qquad (p, q) = 1$$

is not greater than

$$\Gamma^{t+1},$$

where $\Gamma > 0$ is a constant independent of m, and t is the number of distinct prime factors of m.

I may add that thirty years later, in a joint paper with D. J. Lewis of 1961, we could replace this upper bound Γ^{t+1} by the more explicit one

$$\gamma_1 (an)^{\gamma_2 \sqrt{n}} + (\gamma_3 n)^{t+1},$$

where $a = \max(|a_0|, |a_1|, \ldots, |a_n|)$ is the height of $A(x, y)$, and γ_1, γ_2, and γ_3 are three positive absolute constants which can be determined and are not very large. In spite of all the recent progress with Diophantine equations, this seems to be still the best upper bound for the number of solutions.

(**6**) My p–adic generalisation of Thue's theorem appeared in 1933. In the same year, using a method suggested by an unpublished special result by Siegel, I determined an asymptotic formula for the number of solutions of

$$\frac{|A(p, q)|}{Q(p, q)} \leqslant X, \qquad (p, q) = 1$$

as function of X when X tends to infinity. This result implies that if $N(X)$ denotes the number of integers m between $-X$ and $+X$ which can be written in the form $a = A(p, q)$, then a positive constant c_4 exists such that for large X,

$$N(X) \leqslant c_4 X^{2/n}.$$

In 1938, P. Erdös and I succeeded in proving that there is also a positive constant c_5 such that for large X,

$$N(X) \geqslant c_5 X^{2/n}.$$

In later years Ch. Hooley has replaced these two estimates by an asymptotic formula for $N(X)$.

In the special case of cubic binary forms $A(x, y)$ I could prove in 1935 that if the condition $(p, q) = 1$ is omitted, the number of integral solutions of $A(p, q) = m$ can be arbitrarily large for suitable m and even greater than $(\log |m|)^{1/4}$. It seems still to be unknown what the result is when $(p, q) = 1$.

(**7**) After Roth's theorem appeared in 1955, I studied the p-adic implications of his method and in particular proved the following little theorem.

If r and s are integers satisfying

$$2 \leqslant s < r, \qquad (r, s) = 1,$$

if $\varepsilon > 0$ is an arbitrarily small positive constant and k is a sufficiently large positive integer, then

$$\left| \left(\frac{r}{s}\right)^k - \textit{nearest integer} \right| > e^{-\epsilon k}.$$

This result has some interest because it can be applied to Waring's Problem. On combining it with the estimates obtained by a number of mathematicians, it implies the following consequence.

There exists a positive integer k_0 such that if $k \geqslant k_0$ and if

$$g(k) = 2^k + \left[\left(\frac{3}{2}\right)^k \right] - 2,$$

*then every positive integer is the sum of at most $g(k)$ k*th *powers of positive integers, and here $g(k)$ may not be replaced by any smaller number.*

Unfortunately, the proof is non-effective, and we still do not know how large k_0 is.

(**8**) The proofs of Thue's theorem and its improvements are non-effective and provide no methods for obtaining upper bounds for the solutions. The same is true for Siegel's theorem on algebraic curves of 1929.

Fortunately, in more recent years, A. Baker made a breakthrough by studying the approximation properties of logarithms effectively. This allowed him and his students to derive effective bounds for the solutions of $A(p, q) = m$ and of more general Diophantine equations. However, his method has still not yet allowed to give an effective proof of even Thue's inequality $t(\zeta) \leqslant n/2 + 1$.

Also Baker's method can be generalised to the case when not only the absolute value, but also a finite number of different p-adic values are taken into consideration. Of the mathematicians who have worked on such problems I mention in particular Coates and Sprindžuk.

(**9**) I published a number of further papers on Diophantine approximations and equations and in 1961 a small book in which I investigated the p-adic generalisations of Roth's method.

For the case of algebraic curves of genus 1, I gave in 1934 a p-adic generalisation of Siegel's theorem on algebraic curves. I could show that there are at most finitely many rational points (x, y) on such curves such that the greatest prime divisors of the denominators of x and also of y are bounded.

Of a different kind was a generalisation which I gave in 1955 to Siegel's theorem on algebraic curves. I assumed his theorem and applied simple methods from field theory and from the theory of point sets. In this way the following general result could be proved.

Let $F(x, y)$ be an irreducible polynomial with real or complex coefficients such that the algebraic curve

$$F(x, y) = 0$$

is at least of genus 1. *Let further $u_1, u_2, \ldots, u_r, v_1\ v_2, \ldots, v_s$ be finitely many real or complex numbers where both the u_i and the v_j are linearly independent over the rational numbers.*

Then there exist at most finitely many sets of $r + s$ rational integers $x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s$ satisfying the equation

$$F(u_1 x_1 + u_2 x_2 + \cdots + u_r x_r, v_1 y_1 + v_2 y_2 + \cdots v_s y_s) = 0.$$

Consider in particular the case when F has real coefficients and the numbers u_i and v_j are likewise real; let further both r and s be at least 2. Then the points

$$(u_1 x_1 + u_2 x_2 + \cdots + u_r x_r, v_1 y_1 + v_2 y_2 + \cdots + v_s y_s)$$

with integral x_i and y_i lie everywhere dense in the real plane, but only finitely many of these points lie on the curve!

II. Transcendental Numbers

(**10**) A number (real, complex, or p-adic) is said to be transcendental (i.e., over the rational field $\mathbb{Q}$) if it is not algebraic.

The existence of transcendental numbers was first proved by J. Liouville in 1844. As he had proved, the number $t(\zeta)$ of Section 1 is finite for all algebraic numbers ζ. Hence the real number ζ is certainly transcendental if the inequality

$$0 < \left|\zeta - \frac{p}{q}\right| \leqslant q^{-\tau}$$

has for every $\tau > 0$ infinitely many solutions in rational integers p/q such that q tends to infinity. Numbers with this property are called Liouville numbers.

One can prove that the set of all Liouville numbers is non-enumerable, but has the Lebesgue measure 0. A simple example of a Liouville number is given by

$$\sum_{n=1}^{\infty} 2^{-n!}.$$

While all Liouville numbers are transcendental, the converse is of course false. Thus Roth's theorem enables one to show that the number

$$\sum_{n=1}^{\infty} 2^{-3^n}$$

is non-Liouville, but transcendental.

(11) In the eighteenth century, L. Euler conjectured already that both e and π are transcendental. This conjecture was proved to be true a century later, for e by Ch. Hermite in 1873 and for π by F. Lindemann in 1884. Both proofs are based on a system of identities which define rational approximation functions of several exponential functions; these identities were first given by Hermite in his paper of 1873. Some twenty years later Hermite obtained a second system of approximations for exponential functions, but he did not apply this system to problems of transcendency.

During the following years up to the 1920s little progress in the theory of transcendency was made. But then Siegel, in the first part of his paper of 1929, already mentioned in connection with Diophantine equations, introduced a revolutionary method. This method allowed him to study the values at algebraic points of certain classes of entire functions which satisfy linear differential equations with rational functions as coefficients. One of his results was that for rational numbers ν such that $2\nu + 1$ is not an even integer and for every algebraic number $\alpha \neq 0$ the two function values $J_\nu(\alpha)$ and $J'_\nu(\alpha)$ are algebraically independent over $\mathbb{Q}$ and hence are both transcendental. Later, since 1954, A. B. Shidlovski introduced a number of improvements on Siegel's method and obtained very general results on linear differential equations. (His ideas are explained in detail in my book "Lectures on Transcendental Numbers" of 1976).

(12) Only a few years after Siegel's paper of 1929, important progress in the theory of transcendental numbers, but in a different direction, had been made by A. O. Gelfond and Th. Schneider. After an earlier special result by Gelfond, these two mathematicians proved independently and with quite different methods in 1934 that

$$a^b \qquad \text{and} \qquad \frac{\log a}{\log b}$$

are transcendental for algebraic numbers a and b if for the first number a is distinct from 0 and 1 and b is irrational, and if a and b are distinct from 0 and 1 and $\log a/\log b$ is irrational for the second number.

Schneider was particularly successful in generalising his methods. In the following years he obtained important results on the transcendency of elliptic, modular, and Abelian functions. This work can be studied in his book of 1957.

Since 1966, A. Baker has introduced entirely new methods into the theory connected with the algebraic approximations of logarithms. These methods have not only been of fundamental importance for the effective theory of Diophantine equations, but they have also produced very general new classes of transcendental numbers.

(**13**) My own studies of transcendental numbers began about 1926. During a part of that year I was very ill and in bed. To occupy myself, I played with the function

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}$$

and tried to prove that $f(\zeta)$ is irrational for rational ζ satisfying $0 < |\zeta| < 1$. I succeeded and ended by proving that $f(\zeta)$ is transcendental for all algebraic numbers ζ satisfying this inequality.

This result I could later generalise to power series in one or more variables with algebraic coefficients which satisfy a very general type of functional equation. This work was published in three papers of 1929–1930. I mention only one example which is rather pretty.

Let ω be a real quadratic irrationality; let ζ be an algebraic number satisfying $0 < |\zeta| < 1$, and let

$$f(z) = \sum_{n=1}^{\infty} [n\omega]\, z^n,$$

where as usual $[x]$ denotes the integral part of x. Then any finite number of the function values

$$f(\zeta), f'(\zeta), f''(\zeta), \ldots$$

are algebraically independent over $\mathbb{Q}$ and hence all these values are transcendental.

(**14**) Not much later I began to study in detail Hermite's approximation functions of the exponential function and recognised the connection between his two systems of approximations. I further began to understand how these approximation functions could be used for a detailed study of the algebraic approximation of transcendental numbers like e, π, and log 2.

At that time it was well known from the continued fraction for e that e is not a Liouville number, but the analogous problem for π was still unsolved. In two papers that appeared in 1930–1931, I established the following results.

(a) *Let λ be any Liouville number, and let $\omega_1,\ldots,\omega_m$ be finitely many algebraic numbers which are linearly independent over $\mathbb{Q}$. Then the numbers*

$$\lambda, e^{\omega_1},\ldots, e^{\omega_m}$$

are algebraically independent over $\mathbb{Q}$.

(b) *If λ is any Liouville number, then*

$$\lambda \text{ and } \pi$$

are algebraically independent over $\mathbb{Q}$. The same is true for

$$\lambda \text{ and } \log \zeta,$$

where ζ is any rational number distinct from 0 *and* 1.

If the Liouville number λ is omitted in these two statements, we come back to the theorems by Lindemann.

(**15**) The two statements (a) and (b) depended on a new classification of the real and complex numbers which I found in 1927 and in 1935 extended to p-adic numbers. I explain it here because in later years it led to important work by other mathematicians.

For any real or complex polynomial

$$p(z) = p_0 + p_1 z + \cdots + p_m z^m, \qquad \text{where } p_m \neq 0,$$

put

$$\partial(p) = m, \qquad H(p) = \max_{0 \leqslant j \leqslant m} |p_j|, \qquad L(p) = \sum_{j=0}^{m} |p_j|, \qquad \Lambda(p) = 2^{\partial(p)} L(p).$$

Let further ζ be the real or complex number which is to be classified, and let a and n be two positive integral variables. Put

$$\omega_n(a) = \inf |p(\zeta)|$$

where the lower bound is extended over all polynomials $p(z)$ with integral coefficients which satisfy the three conditions

$$\partial(p) \leqslant n, \qquad H(p) \leqslant a, \qquad p(\zeta) \neq 0.$$

It is obvious that $0 < \omega_n(a) \leqslant 1$ and that $\omega_n(a)$ is a non-increasing function of both variables a and n.

Next define two further quantities ω_n and ω as the upper limits

$$\omega_n = \limsup_{a\to\infty} \frac{-\log \omega_n(a)}{\log a} \qquad \text{and} \qquad \omega = \limsup_{n\to\infty} \frac{\omega_n}{n}.$$

Here $0 \leqslant \omega_n \leqslant \infty$ and $0 \leqslant \omega \leqslant \infty$, and ω_n is a non-decreasing function of n.

It may happen that there exist suffixes n for which $\omega_n = \infty$; in this case denote by μ the smallest suffix n with this property, and otherwise put $\mu = \infty$. It is clear from these definitions that always at least one of the two numbers ω and μ has the value ∞.

The number ζ is now called

an A-number	if	$\omega = 0, \mu = \infty$,
an S-number	if	$0 < \omega < \infty, \mu = \infty$,
a T-number	if	$\omega = \mu = \infty$,
a U-number	if	$\omega = \infty, \mu < \infty$.

Correspondingly, we speak of the classes A, S, T, and U.

The class A is identical with that of all (real or complex) algebraic numbers, while the transcendental numbers are distributed among the three classes S, T, and U. The most important property of the classification is that two numbers which are algebraically dependent over $\mathbb{Q}$ always lie in the same class, hence that numbers in different classes are algebraically independent over Q.

The Liouville numbers are those elements of the class U for which $\mu = 1$; this class is thus not empty. There also exist S-numbers, e.g., the number e^a when $a \neq 0$ is an algebraic number.

But until recently, it was not known whether the class T is empty or not. This problem was solved by W. Schmidt in 1968 who proved that there exist T-numbers. However, even today no explicit example of a T-number is known.

I proved already in 1932 that (in the sense of the Lebesgue measure on the real line or in the complex plane) almost all real and complex numbers are S-numbers. After a number of weaker results by others, V. Sprindžuk in 1967 proved that for every positive number ε almost all real numbers satisfy $\omega \leqslant 1 + \varepsilon$ and almost all complex numbers satisfy $\omega \leqslant \frac{1}{2} + \varepsilon$. Since $\omega = 1$ for $\zeta = e$ and $\omega = \frac{1}{2}$ for $\zeta = ie$, Sprindžuk's result is almost best possible.

(**16**) My classification implies that a real or complex number is transcendental if and only if

$$\omega > 0.$$

In more explicit form, the following theorem holds.

The real or complex number ζ is transcendental if and only if, for any positive number τ, there exist a positive integer n and an infinite sequence $\{p_k(z)\}$ of distinct polynomials with integral coefficients such that

$$\partial(p_k) \leqslant n, \qquad 0 < |p_k(\zeta)| < H(p_k)^{-\tau} \qquad (k = 1, 2, 3, \ldots).$$

In the early sixties, I succeeded in replacing this criterion by the following more general test.

The real or complex number ζ is transcendental if and only if there exist an infinite sequence $\{\tau_k\}$ of positive numbers tending to infinity and an infinite sequence $\{p_k(z)\}$ of distinct polynomials with integral coefficients such that

$$0 < |p_k(\zeta)| < \Lambda(p_k)^{-\tau_k}.$$

It may be mentioned that the classification and these tests can be carried over to p-adic numbers with almost no changes.

(**17**) When I began my work on p-adic Diophantine approximations in the late 1920s, I became interested in the problem of the transcendency of the p-adic exponential function

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

This power series converges only for

$$|z|_2 \leqslant \frac{1}{4} \text{ if } p = 2 \qquad \text{and for} \qquad |z|_p \leqslant \frac{1}{p} \text{ if } p \geqslant 3.$$

Therefore the methods used in proving the transcendency of the complex exponential function do not seem to carry over to the p-adic case. However, in 1932, by using some special p-adic properties of this function, I finally succeeded in proving that also the p-adic exponential function is transcendental for all algebraic $z \neq 0$ in the region of convergence.

Similarly, when the proofs by Gelfond and Schneider of the transcendency of a^b and $\log a/\log b$ in the complex case had appeared in 1934, I also could prove their p-adic analogues. For this purpose it was necessary to investigate the relations between the coefficients of a p-adic power series and its zeros.

(**18**) In the summer of 1936 at Groningen in the Netherlands, when I was still working at the University there, a bicycle rider ran into me. As a consequence, the tuberculosis in my right knee bone, which had been dormant for many years, flared up again. It therefore became necessary to undergo several bone operations in 1936 and 1937. This was naturally a very painful period and I was given many morphine injections, although my doctor warned me against their danger.

After a further operation the pains and hence also the injections finally stopped. Then I tried to convince myself that the drug had not damaged my brain by studying the problem of the possible transcendency of the decimal fraction

$$D = 0.123456789101112...$$

in which the successive integers are written one after the other. I found that I could still do mathematics and succeeded in proving the transcendency of both D and of infinitely many more general decimal fractions.

All these decimal fractions were later proved to be normal by P. Erdös; i.e., every possible finite sequence of digits occurs in these fractions with the correct probability. Thus we know now explicit examples of normal transcendental numbers, but it is still unknown whether there are also normal algebraic numbers. A weaker problem which is likewise unsolved asks whether Cantor's set contains any irrational algebraic number.

(**19**) N. I. Feldman, in a number of papers from 1949 to 1963, established new measures of transcendency of e, π, and $\log a$, where a is an algebraic number distinct from 0 and 1. In the 1950s I began to study the same problem, but with a different method, and I finally obtained explicit estimates for the distances of these numbers from rational and algebraic numbers which did not involve any unknown constants. My papers were of 1953 and 1967, and the first paper contained the following two results in particular.

If n is any sufficiently large positive integer, then

$$|e^n - \text{nearest integer}| > n^{-33n}.$$

If p and $q \geqslant 2$ are arbitrary integers, then

$$\left|\pi - \frac{p}{q}\right| > q^{-42}.$$

Recently, and independently, the exponent 42 in the second inequality was improved to less than 21 by E. Wirsing (unpublished) and M. Mignotte

(1974). The latter uses my own method, but with better estimates for the integrals on which the proof is based.

(**20**) I mentioned already Shidlovski's important generalisation of Siegel's theory of linear differential equations. Shortly after the first paper of Shidlovski appeared, I began to lecture on his method and simplified some sections of his proof. Finally, in a paper of 1968, I applied his theorems to the study of the special functions.

$$C_k(z) = \frac{1}{k!}\left(\frac{\partial}{\partial v}\right)^k J_v(z)|_{v=0} \qquad (k = 0, 1, 2, 3).$$

For one week I believed I had succeeded in proving the transcendency of Euler's constant γ as well as that of e^{γ}. But then I found an algebraic identity connecting these functions with their derivatives relative to z which invalidated my proof.

All I could prove finally was that

$$\frac{\pi Y_0(\alpha)}{2J_0(\alpha)} - \left(\log\frac{\alpha}{2} + \gamma\right)$$

is transcendental for all algebraic numbers $\alpha \neq 0$. Here $J_0(z)$ and $Y_0(z)$ denote the Bessel functions of the first and second kinds with suffix 0, respectively.

III. Geometry of Numbers

(**21**) In the last decade of the last century, H. Minkowski introduced into mathematics a new discipline which he called the Geometry of Numbers. He showed its power by many applications to number theory, in particular to the theory of algebraic number fields. His main theory and results he collected in the book Geometrie der Zahlen of which the first part appeared in 1893 and the second part, after his death, in 1910. This work is still of basic importance, although much has been added to the theory during this century.

(**22**) Let

$$\mathbf{x} = (x_1,\dots, x_n), \qquad \mathbf{y} = (y_1,\dots, y_n), \qquad \mathbf{0} = (0,\dots, 0)$$

and

$$\mathbf{x}^{(h)} = (x_{h1},\dots, x_{hn}) \qquad (h = 1, 2,\dots, n)$$

be points in real n-dimensional space $\mathbb{R}^n$. We assume that $n \geqslant 2$ because the case $n=1$ is uninteresting. The point $\mathbf{0}$ is called the origin. No distinction is made between points and vectors, and the sum or difference $\mathbf{x} \pm \mathbf{y}$, the scalar product $c\mathbf{x}$, and the inner product $\mathbf{x} \cdot \mathbf{y}$ are defined by

$$\mathbf{x} \pm \mathbf{y} = (x_1 \pm y_1, \ldots, x_n \pm y_n), \qquad c\mathbf{x} = (cx_1, \ldots, cx_n),$$

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n.$$

Here c and $\mathbf{x} \cdot \mathbf{y}$ are scalars (real numbers).

For $1 \leqslant m \leqslant n$ the points $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ are said to be linearly independent if the equation

$$c_1 \mathbf{x}^{(1)} + \cdots + c_m \mathbf{x}^{(m)} = \mathbf{0}$$

with real coefficients $c_1, \ldots, c_m$ can hold only if

$$c_1 = \cdots = c_m = 0,$$

and they are otherwise called linearly dependent.

If $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent, then the set

$$\Lambda = \{u_1 \mathbf{x}^{(1)} + \cdots + u_n \mathbf{x}^{(n)} \mid u_1, \ldots, u_n = 0, \pm 1, \pm 2, \ldots\}$$

is called a lattice, and the absolute value of the determinant

$$|x_{hk}|_{h,k=1,\ldots,n}$$

is called the determinant of Λ and denoted by $d(\Lambda)$. Always $d(\Lambda) > 0$. Of particular importance is the special lattice Λ_0 which consists of all points with integral coordinates; it has the determinant $d(\Lambda_0) = 1$.

(**23**) A symmetric convex body K is defined as a point set in $\mathbb{R}^n$ with the following properties.

(a) *K is a bounded closed point set.*

(b) *The origin $\mathbf{0}$ is an interior point of K.*

(c) *If $\mathbf{x}$ lies in K, so does the symmetric point $-\mathbf{x} = (-x_1, \ldots, -x_n)$.*

(d) *K is a convex set. This means that if $\mathbf{x}$ and $\mathbf{y}$ belong to K, so do all points*

$$(1-t)\mathbf{x} + t\mathbf{y}, \qquad \text{where } 0 \leqslant t \leqslant 1,$$

of the line segment which joins $\mathbf{x}$ to $\mathbf{y}$.

Important examples of such convex bodies are the n-dimensional cube

$$|x_1| \leqslant c, \ldots, |x_n| \leqslant c,$$

and the n-dimensional sphere

$$x_1^2 + \cdots + x_n^2 \leqslant c^2.$$

The n-fold integral

$$V(K) = \int \underset{K}{\cdots} \int dx_1 \cdots dx_n$$

which exists and has a finite positive value defines the volume of K.

We can associate with K two convex functions

$$F(\mathbf{x}) = F(x_1, \ldots, x_n) \qquad \text{and} \qquad G(\mathbf{y}) = G(y_1, \ldots, y_n).$$

These functions are real valued at all points of $\mathbb{R}^n$ and have the properties

(a) $F(\mathbf{0}) = 0,$ but $F(\mathbf{x}) > 0$ if $\mathbf{x} \neq \mathbf{0}$;

(b) $F(c\mathbf{x}) = |c|\, F(\mathbf{x})$ for all real c;

(c) $F(\mathbf{x} + \mathbf{y}) \leqslant F(\mathbf{x}) + F(\mathbf{y})$,

with analogous properties for $G(\mathbf{y})$. In terms of these functions, K consists exactly of all points $\mathbf{x}$ satisfying

$$F(\mathbf{x}) \leqslant 1,$$

and also of all points $\mathbf{x}$ which satisfy the inequality

$$\mathbf{x} \cdot \mathbf{y} \leqslant G(\mathbf{y}) \qquad \text{for all points } \mathbf{y}.$$

$F(\mathbf{x})$ is called the distance function and $G(\mathbf{y})$ the tac-function of K.

The points $\mathbf{y}$ for which

$$G(\mathbf{y}) \leqslant 1$$

define a second symmetric convex body, K^*, say. This body K^* has $G(\mathbf{y})$ as its distance function and $F(\mathbf{x})$ as its tac-function. In the terminology of classical geometry the two bodies K and K^* are polar to one another relative to the unit sphere

$$x_1^2 + \cdots + x_n^2 = 1.$$

This means that if $\mathbf{x}$ lies on the surface (frontier) of K, then $\mathbf{x} \cdot \mathbf{y} = F(\mathbf{x})$ is a tangential (tac) plane of K^*, and vice versa.

(24) At the beginning of his investigations, Minkowski was interested in the minima of positive definite quadratic forms at points of the special lattice

Λ_0. This problem had been studied half a century earlier by Hermite by means of algebraic methods. Minkowski recognised the connection of the problem to geometry and connected it to the relations between convex bodies and lattices.

About 1891 he found his revolutionary first theorem.

If the volume $V(K)$ of the symmetric convex body K in $\mathbb{R}^n$ is at least equal to 2^n, then K contains at least one point $\mathbf{x} \neq \mathbf{0}$ of the lattice Λ_0.

This theorem allowed many important applications to the theory of algebraic number fields and to Diophantine approximations. Even today the fruitfulness of this theorem in many branches of mathematics is not yet exhausted. Perhaps its best known consequence is Minkowski's theorem on linear forms.

Let $(a_{hk})_{h,k=1,\ldots,n}$ be a real $n \times n$-matrix of determinant 1. Then there exist n integers $x_1,\ldots, x_n$ not all zero such that

$$|a_{h1}x_1 + \cdots + a_{hn}x_n| \leqslant 1 \qquad (h = 1, 2,\ldots, n).$$

Moreover, in all but one of these n inequalities the sign "$\leqslant$" may be replaced by "$<$."

By means of this theorem and its analogues Minkowski proved that the discriminant of every algebraic number field at least of degree 2 is distinct from ± 1.

(**25**) Even deeper and more powerful is Minkowski's second theorem which is concerned with the successive minima of the symmetric convex body K in the lattice Λ_0.

These minima are defined as follows. If c is any positive number, the inequality $F(\mathbf{x}) \leqslant c$ holds for at most finitely many points in Λ_0. It follows that there exists a point $\mathbf{x}^{(1)} \in \Lambda_0$ distinct from $\mathbf{0}$ such that

$$m_1 = F(\mathbf{x}^{(1)})$$

is as small as possible. Further, if k is any of the numbers 2, 3,..., n and if

$$\mathbf{x}^{(h)} \qquad \text{and} \qquad m_h = F(\mathbf{x}^{(h)}) \qquad (h = 1, 2,\ldots, k-1)$$

have already been defined, there exists a point $\mathbf{x}^{(k)} \in \Lambda_0$ such that

$$\mathbf{x}^{(1)}, \mathbf{x}^{(2)},\ldots, \mathbf{x}^{(k)}$$

are linearly independent and

$$m_k = F(\mathbf{x}^{(k)})$$

is a minimum.

The numbers $m_1, \dots, m_n$ are called the successive minima of K in Λ_0. They do not depend on the special choice of the lattice points $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ and satisfy

$$0 < m_1 \leqslant m_2 \leqslant \cdots \leqslant m_n .$$

The second theorem of Minkowski states now the important inequality

$$\frac{2^n}{n!} \leqslant m_1 \cdots m_n V(K) \leqslant 2^n;$$

it obviously implies the first inequality. Here the upper bound 2^n is best possible and is attained for the unit cube. On the other hand, the lower bound $2^n/n!$ is not attained for $n \geqslant 3$, and the best possible value does not seem to be known.

Minkowski's own proof of his second theorem is quite long and involved. He proved it in the second part of his Geometrie der Zahlen which was published only in 1910 after his death.

Minkowski himself made only one application of his second theorem, namely to the study of a certain algorithm for the approximation of numbers in algebraic number fields.

(**26**) Relatively little further progress was made in the geometry of numbers during a number of years after the death of Minkowski. The most important advance was due to H. F. Blichfeldt; in 1914 he generalised and in special cases improved on Minkowski's first theorem. He too used geometrical methods. R. Remak replaced them later by analytical methods depending on integrals (1927), and C. L. Siegel gave a proof in 1935 which applied Fourier series.

New Proofs of Minkowski's second theorem were due to H. Davenport in 1939, to H. Weyl in 1942, and to T. Estermann in 1946.

(**27**) While I was at the University of Groningen from 1934 to 1936, I gave a course on Diophantine approximations which included a detailed discussion of Minkowski's two theorems. Shortly afterwards (1937), I applied Minkowski's second theorem to the global study of algebraic number fields. In this work I considered simultaneously all the valuations in such a field and investigated how the geometry of numbers could be combined with valuation theory and in particular with p-adic numbers. Many years later (1964), an improved method enabled me to establish global estimates for ideal bases in algebraic number fields. These estimates implied particularly simple proofs of the finiteness of the class number and of the existence of independent units.

In 1938, I applied Minkowski's second theorem and derived from it new general theorems. One such theorem states that to every symmetric convex body K there exist n points $X^{(1)},\dots, X^{(n)}$ in Λ_0 of determinant 1 such that

$$\frac{2^n}{n!} \leqslant F(\mathbf{X}^{(1)}) \cdots F(\mathbf{X}^{(n)})\, V(K) \leqslant n!,$$

where the upper bound $n!$ is not in general best possible. I further obtained a similar but less good estimate for the reduced bases of Λ_0 in the sense of Minkowski. Both facts were independently obtained and improved by H. Weyl in 1942.

(**28**) My theorem on the successive minima of polar convex bodies of 1938 proved to be useful in applications. Let $m_1,\dots, m_n$ be the successive minima of K in Λ_0 and let similarly $m_1^*,\dots, m_n^*$ be the successive minima of the polar body K^* in Λ_0. After I had suceeded in proving that the volumes of K and K^* are connected by the inequality

$$4^n(n!)^{-2} \leqslant V(K)\, V(K^*) \leqslant 4^n,$$

where neither of the bounds is best possible, I could deduce from this and from the second theorem of Minkowski applied to both K and K^* that

$$1 \leqslant m_k m_{n-k+1}^* \leqslant (n!)^2 \qquad (k = 1, 2,\dots, n).$$

Here the left-hand inequality is best possible, while the right-hand one can be further improved.

This reciprocity theorem was applied by several mathematicians, in particular by H. Davenport in the study of Dophantine equations in many variables.

Before me, M. Riesz had obtained a similar, but not equivalent theorem on polar convex bodies.

Eighteen years after I had obtained my reciprocity theorem, I found that it was a very special case of a more general theorem connected with the representations of the real linear group (1956). A little earlier, I had already studied the special case of this theorem dealing with compound convex bodies.

Such compound bodies are defined a follows. Let $2 \leqslant p \leqslant n-1$, and let $\mathbf{x}^{(1)},\dots, \mathbf{x}^{(p)}$ be any p point in R^n; further put

$$N = \binom{n}{p} \qquad \text{and} \qquad P = \binom{n-1}{p-1}.$$

The $p \times n$ matrix formed by the coordinates of $\mathbf{x}^{(1)},\dots, \mathbf{x}^{(p)}$ contains N

independent minors of order p which we denote in some fixed, but arbitrary, order by $X_1,\dots, X_N$. The point

$$\boldsymbol{X} = (X_1,\dots, X_N)$$

in N-dimensional space $\mathbb{R}^N$ defines the pth compound of $\mathbf{x}^{(1)},\dots, \mathbf{x}^{(p)}$.

Let now $\mathbf{x}^{(1)},\dots, \mathbf{x}^{(p)}$ run independently over the symmetric convex body K. Then $\mathbf{X}$ describes a certain symmetric closed bounded point set S in $\mathbb{R}^N$ which lies on the so-called Grassmann manifold and is in general not convex. Its convex closure, K^p, say, is, however, a symmetric convex body in $\mathbb{R}^N$.

I could show that the N-dimensional volume $V(K^p)$ of K^p is connected with the n-dimensional volume $V(K)$ of K by the inequalities

$$C_1 \leqslant V(K^{(p)})\, V(K)^{-p} \leqslant C_2,$$

where C_1 and C_2 just as later c_1 and c_2 denote positive constants which depend only on n and p, but not on the special convex bodies K and $K^{(p)}$.

This property of the volumes implies now again a set of inequalities for the N successive minima, $m_1^p,\dots, m_N^p$, say, of K^p in the lattice Λ_0^p of all points in $\mathbb{R}^N$ with integral coordinates relative to the successive minima $m_1,\dots, m_n$ of K in Λ_0. In order to formulate these inequalities arrange the N products

$$m_{i_1} m_{i_2} \cdots m_{i_p}, \qquad \text{where} \quad 1 \leqslant i_1 < i_2 < \cdots < i_p,$$

in order of increasing size and then denote them by $M_1, M_2,\dots, M_N$, respectively. Thus

$$0 < M_1 \leqslant M_2 \leqslant \cdots \leqslant M_N.$$

The wanted inequalities take then the simple form

$$c_1 M_k \leqslant m_k^p \leqslant c_2 M_k \qquad (k = 1, 2,\dots, N).$$

This theorem has recently found applications. It forms one of the tools in the work by W. Schmidt (1970) on the simultaneous approximations of several algebraic numbers by rational numbers, a generalisation of Roth's theorem.

(29) All the results mentioned so far concern the geometry of numbers of convex bodies. However, already Minkowski himself began the study of the relations between lattices and general point sets. He stated without proof one property of this kind. His conjecture was first proved by E. Hlawka in 1944. Then, in 1945, C. L. Siegel obtained Minkowski's conjecture from a deep theorem of his on the Haar measure in the space of lattices in $\mathbb{R}^n$.

(30) The real revolution in the geometry of numbers began, however, already several years earlier. Before I give details on it, it is convenient to introduce a number of notations.

Let S be an arbitrary bounded or unbounded closed point set in $\mathbb{R}^n$ and let further Λ be a variable lattice in this space. The lattice is said to be S-admissible if none of its points $\mathbf{x} \neq \mathbf{0}$ is an interior point of S. If now S has no admissible lattices, then put

$$\Delta(S) = \infty;$$

otherwise define $\Delta(S)$ by

$$\Delta(S) = \inf d(\Lambda),$$

where the lower bound is extended over all S-admissible lattices. An S-admissible lattice Λ which satisfies the equation

$$d(\Lambda) = \Delta(S)$$

is called a critical lattice of S. It is clear from this definition that such critical lattices cannot exist unless $0 < \Delta(S) < \infty$.

Of particular importance are the symmetric star bodies in R^n which are defined by the following three properties.

(a) *S is a bounded or unbounded closed point set in $\mathbb{R}^n$.*

(b) *If $\mathbf{x}$ lies in S, then so do all points of the line segment*

$$t\mathbf{x}, \qquad \textit{where} \quad -1 \leqslant t \leqslant 1.$$

(c) *$\mathbf{0}$ is an interior point of S.*

In the case of a star body, the lattice determinant $\Delta(S)$ cannot vanish, but may have any positive value or be equal to $+\infty$.

Star bodies in 2 dimensions are called star domains.

(31) The modern trend in the geometry of numbers began in 1939 with several papers by H. Davenport. By means of algebraic considerations he determined $\Delta(S)$ and the critical lattices for the two unbounded star bodies

$$|x_1 x_2 x_3| \leqslant 1 \qquad \text{and} \quad |x_1(x_2^2 + x_3^2)| \leqslant 1$$

in $\mathbb{R}^3$. This was the first time that $\Delta(S)$ had been obtained for a non-convex point set in $\mathbb{R}^3$.

Next, in 1940, L. J. Mordell succeeded in reducing these two three-dimensional problems to problems in $\mathbb{R}^2$ which he could then solve by means of geometrical considerations. This geometrical method for finding $\Delta(S)$ and the critical lattices he later entended to a large class of star domains. All this work made essential use of Minkowski's first theorem.

These investigations by Davenport and Mordell on non-convex point sets were continued by them in the following years, and they were soon joined by their students.

(32) I myself began to work on non-convex point sets, at first in $\mathbb{R}^2$, in 1942, by establishing a general (but not very practical) method for obtaining all the critical lattices of a star domain bounded by finitely many analytical arcs. Not much later I began to study the general lattice properties of star bodies in $\mathbb{R}^n$ for $n \geqslant 2$ and finally also of arbitrary point sets in this space.

I tried to obtain general laws rather than deal with special examples. I finally arrived at a very useful compactness theorem for lattices from which such laws could be derived.

The compactness theorem can be formulated as follows.

A sequence $\{\Lambda_k\}$ of lattices in $\mathbb{R}^n$ is said to converge to a further lattice Λ if, however large the number $r > 0$ is chosen, all the points of all the lattices Λ_k inside the sphere

$$x_1^2 + \cdots x_n^2 \leqslant r^2$$

tend to the points of Λ. This property is satisfied exactly if suitably chosen bases of the lattices Λ_k tend term by term to a basis of Λ.

The sequence $\{\Lambda_k\}$ is further said to be bounded if all the determinants $d(\Lambda_k)$ are bounded and if, moreover, there exists a neighbourhood of the origin $\mathbf{0}$ of $\mathbb{R}^n$ which contains no point $\mathbf{x} \neq \mathbf{0}$ of any lattice Λ_k.

The compactness theorem is now as follows.

Every bounded sequence of lattices contains a convergent subsequence. This theorem allows many applications. As an example I prove the following theorem.

Every star body S with finite $\Delta(S)$ has at least one critical lattice.

Proof. By the definition of $\Delta(S)$, there exists an infinite sequence $\{\Lambda_k\}$ of S-admissible lattices Λ_k such that

$$\lim_{k \to \infty} d(\Lambda_k) = \Delta(S).$$

By the definition of a star body, S contains a neighbourhood of $\mathbf{0}$. Since the lattices Λ_k are S-admissible, none of their points distinct from $\mathbf{0}$ can lie in this neighbourhood; also their determinants are bounded. Hence they form a bounded sequence. There exists then an infinite subsequence $\{\Lambda_{k_j}\}$ of $\{\Lambda_k\}$ with $k_j \to \infty$ which converges to a certain lattice Λ. Here

$$d(\Lambda) = \lim_{j\to\infty} d(\Lambda_{k_j}) = \Delta(S).$$

This lattice Λ is critical. Otherwise there would exist a point $\mathbf{x} \neq \mathbf{0}$ of Λ which is an interior point of S. But then, for all sufficiently large j, also Λ_{k_j} contains an interior point of S, contrary to the definition of admissible lattices.

It may be remarked that the set of all critical lattices of a star body can be finite or infinite, enumerable or non-enumerable.

The compactness theorem also allows us to obtain necessary and sufficient conditions for arbitrary point sets in R^n to have critical lattices.

While a critical lattice of S does not contain interior points $\boldsymbol{x} \neq \mathbf{0}$ of S, it may contain frontier points of this set. However, already the simple star domain

$$x_1^2(x_1^2 + x_2^2) \leqslant 1$$

has no frontier points on any one of its infinitely many critical lattices.

In my papers of 1946 and later I made many applications of the compactness theorem, and I was soon followed by other mathematicians, in particular by Davenport, J. W. S. Cassels, and C. A. Rogers in such applications.

On the remaining pages of these notes I shall collect references. These go to about 1970. However, the list of my own publications has been extended to 1981.

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(33) No attempt has been made in the following list at being complete. I have tried to include those papers which influenced my own research.

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