On the analytic solution of certain functional and difference equations

By K. Mahler, F.R.S.

Mathematics Department, Research School of Physical Sciences,
Australian National University, Canberra, A.C.T. 2601, Australia

Let $P(u, v)$ be an irreducible polynomial with complex coefficients and let $q \geqslant 2$ be an integer. We establish the necessary and sufficient conditions under which the functional equation

$$P(f(z), f(z^q)) = 0, \tag{F}$$

has a non-constant analytic solution that is either regular in a neighbourhood of the point $z = 0$ or has a pole at this point (theorem 1).

By a simple change of variable, the difference equation

$$P(F(Z), F(Z+1)) = 0, \tag{D}$$

can be proved under the same restrictions to have a non-constant solution of the form

$$F(Z) = \sum_{j=I}^{\infty} f_j \, \mathrm{e}^{-jq^Z},$$

which is regular in the strip

$$\mathrm{Re}\, Z \geqslant X_0, \quad |\mathrm{Im}\, Z| < \pi/2 \ln q,$$

if X_0 is sufficiently large (theorem 2).

1.

Let

$$f(z) = \sum_{j=0}^{\infty} f_j \, z^j,$$

be a formal power series in z with complex coefficients f_j. If $f(z)$ consists only of the constant term f_0, then $f(z)$ is called a *forbidden* series; otherwise $f(z)$ is said to be *admissible*. The set of all admissible series is denoted by X.

Throughout this paper, q denotes an integer at least 2. If the power series $f(z)$ is forbidden, so is $f(z^q)$, and vice versa. Obviously all power series satisfying the functional equation

$$f(z^q) = f(z),$$

are forbidden.

To every admissible power series $f(z)$ there exists a smallest suffix $r \geqslant 1$ such that $f_r \neq 0$. By putting

$$\phi_j = f_{j+r}/f_r \quad (j = 0, 1, 2, \ldots),$$

so that $\phi_0 = 1$, $f(z)$ assumes the normed form

$$f(z) = f_0 + f_r z^r \left(1 + \sum_{j=1}^{\infty} \phi_j z^j\right), \quad \text{where} \quad f_r \neq 0. \tag{I}$$

If we wish to show that $f(z)$ belongs to the suffix $r \geqslant 1$, then we write

$$f(z) = f(z; r).$$

2.

Next denote by Y the set of all irreducible polynomials

$$P(u, v) = \sum_{h=0}^{m} \sum_{k=0}^{m} P_{hk} x^h y^k,$$

with complex coefficients P_{hk}, of the exact degree $m \geqslant 1$ in u and the exact degree $n \geqslant 1$ in v. If also $Q(u,v) \in Y$, and if there is a complex constant $c \neq 0$ such that $Q(u,v) = cP(u,v)$, then $P(u,v)$ and $Q(u,v)$ are said to be *associated*.

Definition. Let $f(z) \in X$ and $P(u,v) \in Y$. If the formal power series $f(z)$ satisfies the formal functional equation

$$P(f(z), f(z^q)) = 0, \tag{F}$$

then $f(z)$ is called a q-series and said to belong to $P(u,v)$, and conversely $P(u,v)$ is said to belong to $f(z)$.

It will immediately be proved that there is essentially only one polynomial $P(u,v)$ belonging to the q-series $f(z)$. The later results will, however, show that more than one q-series may belong to a given polynomial $P(u,v) \in Y$, although their number is always finite.

LEMMA 1. *If the q-series $f(z)$ belongs to both $P(u,v)$ and $Q(u,v)$, then these two polynomials are associated.*

Proof. Let $R(u)$ be the resultant of $P(u,v)$ and $Q(u,v)$ with respect to v. Then

$$R(u) = P(u,v) P^*(u,v) + Q(u,v) Q^*(u,v),$$

with certain polynomials $P^*(u,v)$ and $Q^*(u,v)$. If $R(u)$ is not identically zero, then the functional equation (F) and its analogue for $Q(u,v)$ show that

$$R(f(z)) = 0,$$

hence that $f(z)$ is a constant and therefore forbidden, contrary to the hypothesis. Therefore $R(u)$ vanishes identically. But then the two polynomials $P(u,v)$ and $Q(u,v)$ have a non-zero common factor and so, by their irreducibility, are associated.

LEMMA 2. *Let the polynomial $P(u,v) \in Y$ not be associated with the polynomial $u - v$. Then $P(u,u)$ is not identically zero.*

Proof. Put

$$P_h(u,v) = (1/h!)\,(\partial/\partial v)^h\,P(u,v) \quad (h = 1, 2, ..., m),$$

so that

$$P_0(u,v) = P(u,v).$$

By Taylor's formula,

$$P(u,v) = P(u, u + (v-u)) = \sum_{h=0}^{m} P_h(u,u)\,(v-u)^h.$$

It follows that if $P(u,u)$ vanishes identically, then $P(u,v)$ is divisible by, and therefore associated with, the polynomial $u-v$.

LEMMA 3. *Let $f(z)$ be a q-series which belongs to $P(u,v)$. Then the constant term f_0 of $f(z)$ satisfies the equation*

$$P(f_0,f_0) = 0. \tag{C_1}$$

Proof. This assertion follows at once from the functional equation (F) on substituting $z = 0$.

Thus if $P(u,v)$ is not associated with $u-v$, then the constant term f_0 of any q-series belonging to $P(u,v) \in Y$ has at most finitely many possible values. On the other hand, if $P(u,v) = c(u-v)$, then it is easily seen that there does not exist any q-series belonging to $P(u,v)$.

<div align="center">3.</div>

Denote by $P(u,v)$ any polynomial in Y which is not associated with $u-v$, and by $f(z)$ any q-series belonging to $P(u,v)$. We know already that this requires the equation (C_1), but this condition alone is not sufficient, and we shall establish three further conditions which have to be satisfied.

For this purpose consider simultaneously with the functional equation (F) the algebraic equation

$$P(u,v) = 0. \tag{A}$$

Since $P(u,v)$ is irreducible and has the exact degree $m \geqslant 1$ in u, this equation defines u as an m-valued algebraic function

$$u = U(v)$$

of v which is not a constant. The equation (C_1) implies that there exists a branch $U_0(v)$ of $U(v)$ with the property

$$U_0(f_0) = f_0,$$

where f_0 is again the constant term of $f(z)$.

By the theory of algebraic functions it may be assumed that this branch $U_0(v)$ has in a neighbourhood of $v = f_0$ the power series

$$U_0(v) = f_0 + \sum_{l=1}^{\infty} p_l(v - f_0)^{l/a},$$

with certain complex coefficients p_l not all zero; here a is a positive integer which does not exceed the degree m. Denote by $b \geqslant 1$ the smallest suffix such that

$$p_b \neq 0.$$

Such a suffix exists because $U(v)$ and hence also $U_0(v)$ is not a constant. The series for $U_0(v)$ has thus the explicit form

$$U_0(v) = f_0 + \sum_{l=b}^{\infty} p_l(v-f_0)^{l/a}.$$

Remark. If ω is any ath root of unity, than also the series

$$U_0(v|\omega) = f_0 + \sum_{l=b}^{\infty} p_l\omega^l(v-f_0)^{l/a}$$

represents a branch of $U(v)$. On writing

$$p_l \quad \text{for} \quad p_l\omega^l,$$

$U_0(v|\omega)$ takes again the form $U_0(v)$. It therefore suffices to consider the latter branch of $U(v)$.

If the integer a is smaller than m, then there are further branches of $U(v)$.

4.

Consider now any q-series

$$f(z) = f(z; r),$$

of the form (I) that belongs to $P(u,v) \in Y$. By the functional equation (F) the pair of formal power series

$$u = f(z), \quad v = f(z^q),$$

satisfies the algebraic equation (A) and hence may be assumed to lie on the branch $U_0(v)$ of $U(v)$. It follows then from (F) that also

$$f(z) = f_0 + \sum_{l=b}^{\infty} p_l(f(z^q)-f_0)^{l/a}, \tag{F_1}$$

a formal identity. Here, by (I),

$$f(z) = f_0 + f_r z^r \left(1 + \sum_{j=1}^{\infty} \phi_j z^j\right),$$

$$f(z^q) = f_0 + f_r z^{qr} \left(1 + \sum_{j=1}^{\infty} \phi_j z^{qj}\right),$$

so that (F_1) is equivalent to

$$f_r z^r \left(1 + \sum_{j=1}^{\infty} \phi_j z^j\right) = \sum_{l=b}^{\infty} p_l f_r^{l/a} z^{lqr/a} \left(1 + \sum_{j=1}^{\infty} \phi_j z^{jq}\right)^{l/a}. \tag{F_2}$$

Here, by the binomial theorem, the last factor may be identified with the series

$$\left(1 + \sum_{j=1}^{\infty} \phi_j z^{jq}\right)^{l/a} = 1 + \sum_{k=1}^{\infty} \binom{l/a}{k} \left(\sum_{j=1}^{\infty} \phi_j z^{jq}\right)^k,$$

which consists of terms in integral powers of z^q. Hence (F_2) implies that

$$f_r z^r \left(1 + \sum_{j=1}^{\infty} \phi_j z^j\right) = \sum_{l=b}^{\infty} p_l f_r^{l/a} z^{lqr/a} \left[1 + \sum_{k=1}^{\infty} \binom{l/a}{k} \left(\sum_{j=1}^{\infty} \phi_j z^{jq}\right)^k\right]. \qquad (F_3)$$

Since this is an identity, equal powers of z on its two sides have the same coefficients. Moreover, necessarily the terms of lowest degree in z on the left and the right sides are the same, thus

$$f_r z^r = p_b f_r^{b/a} z^{bqr/a}.$$

This equation implies that

$$f_r = p_b^{a/(a-b)}, \qquad (C_2)$$

and

$$a = bq, \quad \text{hence} \quad 2 \leqslant q \leqslant bq = a \leqslant m. \qquad (C_3)$$

There is one further necessary condition which can be best formulated in terms of the following notation.

Definition. Denote by R the smallest positive integer such that lqR/a is an integer whenever $l > b$ and $p_l \neq 0$.

It is obvious that such an integer R exists and is a divisor of a.

The fourth necessary condition takes now the simple form

$$r = dR, \quad \text{where} \quad d \text{ is a positive integer.} \qquad (C_4)$$

Proof of (C_4). If r is not of this form, then there exists a smallest suffix $\lambda > b$ such that $p_\lambda \neq 0$ and that $\lambda qr/a$ is not an integer. Now the right side of the identity (F_3) contains the non-vanishing term

$$p_\lambda f_r^{\lambda/a} z^{\lambda qr/a},$$

where the exponent of z is fractional, and there is no other term on this right side involving the same power of z. Since the left side of (F_3) contains only terms in integral powers of z, a contradiction arises. The following result has thus been proved.

LEMMA 4. *Let $f(z) = f(z; r)$ be a q-series belonging to $P(u, v) \in Y$. Then the four conditions* (C_1), (C_2), (C_3), *and* (C_4) *are satisfied.*

COROLLARY. *The branch $U_0(v)$ has the explicit form $U_0(v) = f_0 + p_b(v - f_0)^{1/q}$ plus terms in higher powers of $v - f_0$, for by* (C_3), $b/a = 1/q$.

5.

Lemma 4 has the following converse.

LEMMA 5. *Let $P(u, v) \in Y$, and let f_0, r, f_r, and q satisfy the conditions* (C_1), (C_2), (C_3), *and* (C_4). *Then there exists a unique q-series*

$$f(z) = f(z; r) = f_0 + f_r z^r \left(1 + \sum_{j=1}^{\infty} \phi_j z^j\right)$$

satisfying the formal functional equation (F_3).

Proof. Divide both sides of (F$_3$) by the factor

$$f_r z^r = p_b f_r^{b/a} z^{bqr/a},$$

and replace the suffix l on the right side by $b+l$. On putting

$$\pi_l = p_{b+l} f_r^{(b+l)/a} f_r^{-1} = p_{b+l} f_r^{(b+l)/a} p_b^{-1} f_r^{-b/a} = (p_{b+l}/p_b) f_r^{l/a} \quad (l = 0, 1, 2, \ldots),$$

so that

$$\pi_0 = 1,$$

the identity (F$_3$) takes the equivalent form

$$1 + \sum_{j=1}^{\infty} \phi_j z^j = \sum_{l=0}^{\infty} \pi_l z^{lqr/a} \left[1 + \sum_{k=1}^{\infty} \binom{(b+l)/a}{k} \sum_{j_1=1}^{\infty} \cdots \sum_{j_k=1}^{\infty} \phi_{j_1} \cdots \phi_{j_k} z^{(j_1+\cdots+j_k)q} \right]. \quad \text{(F}_4\text{)}$$

Here it is convenient to define π_l also for fractional l by putting it then equal to 0.

By means of (F$_4$) the coefficients ϕ_j of $f(z)$ can now be defined for the successive suffixes j. For on comparing the coefficients of z^j on both sides of (F$_4$), we obtain the system of equations

$$\phi_j = \pi_{aj/qr} + \sum_l \sum_k \sum_{j_1,\ldots,j_k} \pi_l \binom{(b+l)/a}{k} \phi_{j_1} \cdots \phi_{j_k} \quad (j = 1, 2, 3, \ldots), \quad \text{(E)}$$

where the summation extends over all sets of suffixes $l, k, j_1, \ldots, j_k$ for which

$$l \geqslant 1, \quad k \geqslant 1, \quad j_1 \geqslant 1, \ldots, j_k \geqslant 1, \quad lqr/a + (j_1 + \cdots + j_k)q = j, \quad (1)$$

hence also

$$1 \leqslant l < aj/qr, \quad 0 \leqslant k \leqslant [j/q], \quad k \leqslant j_1 + \cdots + j_k \leqslant [j/q]. \quad (2)$$

From the equations (E), each coefficient ϕ_j can be expressed as a polynomial in $\phi_1, \phi_2, \ldots, \phi_{j-1}$ and in fact in $\phi_1, \phi_2, \ldots, \phi_{[j/q]-1}$. Hence these coefficients are uniquely determined, as was to be proved.

COROLLARY. *For every positive integer d there exists a unique q-series $f(z; dR)$ satisfying* (F$_1$). *Moreover,*

$$f(z; dR) = f(z^d; R). \quad (3)$$

For it is clear that if $f(z)$ is a solution of (F$_1$), then $f(z^d)$ also is a solution. The assertion (3) is thus a direct consequence of the uniqueness of the solution of given suffix dR.

<div align="center">6.</div>

The q-series $f(z)$ given by lemma 5 is a formal power series. We shall now prove the surprising result that this power series has in fact a region of convergence and hence in this region defines an analytic function of z.

The proof uses the following almost trivial lemma on binomial coefficients.

LEMMA 6. *Let $x > 0$ be a real number and $k \geqslant 1$ an integer. Then*

$$\left| \binom{x}{k} \right| < e^{x+1}.$$

Proof. From its definition as a binomial coefficient,

$$\binom{x}{k} = \left(\frac{x+1}{1} - 1\right)\left(\frac{x+1}{2} - 1\right) \cdots \left(\frac{x+1}{k} - 1\right). \tag{4}$$

If, firstly, $1 \leqslant k \leqslant x+1$, then all factors on the right side are non-negative and therefore

$$0 < \binom{x}{k} < \frac{x+1}{1}\frac{x+1}{2} \cdots \frac{x+1}{k} = \frac{(x+1)^k}{k!} < \sum_{k=0}^{\infty} \frac{(x+1)^k}{k!} = e^{x+1}.$$

If, secondly, $k > x+1$, then the additional factors $(x+1)/h - 1$ in (4) where $x+1 \leqslant h \leqslant k$ all lie in the interval between -1 and 0, thus have an absolute value at most 1, so that the assertion holds also now.

<h2 style="text-align:center">7.</h2>

By the theory of algebraic functions, the power series

$$U_0(v) = f_0 + \sum_{l=b}^{\infty} p_l(v-f_0)^{l/a},$$

for the branch $U_0(v)$ of $U(v)$ has a positive radius of convergence. By the definition of the coefficients π_l in §5, this imples that there exists a constant

$$\Gamma \geqslant 2$$

such that

$$|\pi_l| \leqslant \Gamma^l \quad \text{for} \quad l = 1, 2, 3, \ldots. \tag{5}$$

Now put

$$c = \max\left(|\phi_1|, 5m\,e^7\Gamma^{\frac{1}{2}}\right).$$

We introduce the following notation.

Definition. Let J be any integer at least 2. The sequence of coefficients $\{\phi_j\}$ of $f(z)$ is said to have the property $[J]$ if

$$|\phi_j| \leqslant c^j \quad \text{for} \quad j = 1, 2, \ldots, J-1.$$

It is obvious that $\{\phi_j\}$ has the property $[2]$. In addition, we shall prove the following result.

LEMMA 7. *If $\{\phi_j\}$ has the property $[J]$, then it has also the property $[J+1]$.*

COROLLARY. *The sequence $\{\phi_j\}$ has the property $[J]$ for every integer $J \geqslant 2$, and therefore*

$$|\phi_j| \leqslant c^j \quad \text{for} \quad j = 1, 2, 3, \ldots. \tag{6}$$

<h2 style="text-align:center">8. PROOF OF LEMMA 7</h2>

We apply the equation (E) for ϕ_j in the special case when $j = J$ and give rough upper estimates for the terms on its right side and for the number of such terms.

Firstly, by (5), for all suffices l satisfying $1 \leqslant l \leqslant aJ/qr$

$$|\pi_l| \leqslant \Gamma^{aJ/qr} \leqslant \Gamma^{\frac{1}{2}mJ}$$

because $q \geqslant 2$, $r \geqslant 1$, and $1 \leqslant a \leqslant m$. In particular,

$$|\pi_{aJ/qr}| \leqslant \Gamma^{aJ/qr} \leqslant \Gamma^{\frac{1}{2}mJ}$$

independent of whether the suffix aJ/qr is an integer or not.

Secondly, by lemma 6,

$$\left|\binom{(b+l)/a}{k}\right| \leqslant e^{(a+b+l)/a} \leqslant e^{(a+a/q+aJ/qr)/a} \leqslant e^{1+\frac{1}{2}+\frac{1}{2}J} \leqslant e^{2J}$$

because $a = bq$, $q \geqslant 2$, $J \geqslant 2$, and $1 \leqslant l \leqslant aJ/qr$.

Thirdly, by the assumption [J],

$$|\phi_{j_1} \dots \phi_{j_k}| \leqslant c^{j_1 + \dots + j_k} \leqslant c^{[J/q]} \leqslant c^{\frac{1}{2}J},$$

because by (2) the suffixes $j_1, \dots, j_k$ satisfy the inequality

$$j_1 + \dots + j_k \leqslant [J/q] \leqslant \tfrac{1}{2}J.$$

Finally, by the upper bounds (2) for the suffixes applied when $j = J$, the suffix l in (E) has at most $aJ/qr \leqslant \frac{1}{2}mJ$ possibilities, the suffix k at most $[J/q] \leqslant \frac{1}{2}J$, and the set of k suffixes $j_1, \dots, j_k$ at most

$$\binom{[J/q] + k - 1}{k} \leqslant \binom{(\frac{1}{2}J) + (\frac{1}{2}J)}{k} \leqslant e^{J+1}.$$

By these estimates, it follows now from (E) that

$$|\phi_J| \leqslant \Gamma^{\frac{1}{2}mJ} + \tfrac{1}{2}mJ \times \tfrac{1}{2}J \times e^{J+1} \times \Gamma^{\frac{1}{2}mJ} \times e^{2J} \times c^{\frac{1}{2}J} \leqslant mJ^2 e^{3J+1}\Gamma^{\frac{1}{2}mJ}c^{\frac{1}{2}J}.$$

Here

$$mJ^2 e^{3J+1}\Gamma^{\frac{1}{2}mJ}c^{\frac{1}{2}J} \leqslant c^J \tag{7}$$

provided that

$$c \geqslant (mJ^2)^{2/J} e^{6+(2/J)}\Gamma^{\frac{1}{2}}.$$

Now $J \geqslant 2$ and therefore

$$J^{4/J} \leqslant 5,$$

while by the definition of c,

$$c \geqslant 5m\,e^7\Gamma^{\frac{1}{2}}.$$

The inequality (7) is then satisfied, and it follows that $|\phi_J| \leqslant c^J$, as was to be proved.

The following result has thus been established.

THEOREM 1. *Let $P(u,v)$ be an irreducible polynomial with complex coefficients of the exact degrees $m \geqslant 1$ and $n \geqslant 1$ in u and v, respectively; assume that $P(u,v)$ is not a constant multiple of $u - v$. Let $q \geqslant 2$ and $r \geqslant 1$ be integers and let*

$$f(z) = f_0 + f_r z^r\left(1 + \sum_{j=1}^{\infty}\phi_j z^j\right), \quad \text{where} \quad f_r \neq 0, \tag{I}$$

be a formal power series with complex coefficients satisfying the formal functional equation

$$P(f(z), f(z^q)) = 0. \tag{F}$$

Then there exist two integers $a \geqslant 1$ and $b \geqslant 1$ and a branch

$$U_0(v) = f_0 + \sum_{l=b}^{\infty} p_l(v - f_0)^{l/a}, \quad \text{where} \quad p_b \neq 0,$$

of the algebraic function $u = U(v)$ defined by

$$P(U(v), v) = 0$$

such that $f(z)$ also satisfies the formal functional equation

$$f(z) = f_0 + \sum_{l=b}^{\infty} p_l(f(z^q) - f_0)^{l/a}, \tag{F_1}$$

if and only if the following four conditions hold.

$$P(f_0, f_0) = 0. \tag{C_1}$$

$$f_r = p_b^{a/(a-b)} = p_b^{q/(q-1)}. \tag{C_2}$$

$$2 \leqslant q \leqslant bq = a \leqslant m. \tag{C_3}$$

The suffix r in (I) has the property that lqr/a is an integer

$$\text{whenever } l > b \text{ and } p_l \neq 0. \tag{C_4}$$

When these four conditions (C) are satisfied, then there exists to r exactly one power series (I) satisfying both (F) and (F_1). Moreover, this power series converges in a neighbourhood of $z = 0$ and here defines an analytic function $f(z)$.

COROLLARY. *A repeated application of (F) allows us to express $f(z^{q^n})$ for $n = 1, 2, 3, \ldots$ as an algebraic function of $f(z)$. Since*

$$z^{q^n} \to 0 \quad \text{if} \quad |z| < 1 \quad \text{and} \quad n \to \infty,$$

it follows that $f(z)$ can be continued as an analytic function into the whole unit disc

$$\text{D: } |z| < 1.$$

However, this function $f(z)$ may have an infinite sequence of poles and algebraic branch points and may be multi-valued.

Examples show that q-functions $f(z)$, as given by theorem 1, may be entire or meromorphic functions in the whole z-plane. But there also are examples in which $f(z)$ cannot be continued beyond the unit circle $|z| = 1$, but still is regular and single-valued in D and continuous on $|z| = 1$; or in which $f(z)$ is infinitely many valued in D with an infinite sequence of algebraic branch points in D tending to every point of $|z| = 1$.

Theorem 1 can be extended to functions $f(z)$ defined by Laurent series

$$f(z) = \sum_{j=I}^{\infty} f_j z^j,$$

where I is any *negative* integer. For then $f(z)^{-1}$ is again a power series, and if $f(z)$ belongs to the polynomial $P(u, v)$, then $f(z)^{-1}$ belongs to the polynomial

$$P^*(u, v) = u^m v^n P(u^{-1}, v^{-1}),$$

and this new polynomial evidently is also irreducible and is not a constant multiple of $u - v$. Theorem 1 can thus be applied to $f(z)^{-1}$.

<div style="text-align:center">9.</div>

As an application let us consider the power series and the Laurent series which satisfy a functional equation

$$f(z^q) = P_0 + P_1 f(z) + \ldots + P_m f(z)^m, \tag{8}$$

where as before q is an integer at least 2, m also is at least 2, and $P_0, P_1, \ldots, P_m$ are complex constants where

$$P_m \neq 0.$$

Firstly let $f(z)$ be again a power series of the form (I). Then by (C$_1$),

$$f_0 = P_0 + P_1 f_0 + \ldots + P_m f_0^m.$$

This equation has m roots, not necessarily all distinct; denote by f_0 any one of these roots. It is convenient to replace $f(z)$ by the new series

$$g(z) = f(z) - f_0 = f_r z^r \left(1 + \sum_{j=1}^{\infty} \phi_j z^j \right), \quad \text{where} \quad f_r \neq 0.$$

The functional equation (8) takes then the form

$$g(z^q) = Q_1 g(z) + Q_2 g(z)^2 + \ldots + Q_m g(z)^m, \tag{9}$$

where

$$Q_m = P_m \neq 0.$$

With the functional equation (9) we associate again the algebraic equation

$$Q(u, v) = 0, \quad \text{where} \quad Q(u, v) = Q_1 u + Q_2 u^2 + \ldots + Q_m u^m - v.$$

Let s, with $1 \leqslant s \leqslant m$, be the suffix for which

$$Q_1 = Q_2 = \ldots = Q_{s-1} = 0, \quad \text{but} \quad Q_s \neq 0.$$

Then the algebraic function $u = U(v)$ defined by

$$Q(U(v), v) = 0,$$

has a branch of the form

$$U_0(v) = Q_1^{-1/s} v^{1/s} + \text{terms in higher powers of } v.$$

This requires by the condition (C$_3$) that

$$s = q \geqslant 2, \quad \text{hence} \quad Q_1 = 0,$$

and in the former notation $a = q = s$, $b = 1$.

Hence, if r is chosen so that also the condition (C$_4$) is satisfied, say as a multiple of $s = q$, then the functional equation (8) has a solution $f(z)$ of the form (I) which is regular in a neighbourhood of $z = 0$, provided only that $s \geqslant 2$. This requires that

$$f_0 = P_0 + P_1 f_0 + \ldots + P_m f_0^m = 0 \quad \text{and} \quad P_1 + 2P_2 + \ldots + mP_m = 0.$$

It is clear that if these two equations are not satisfied, then the only analytic solutions of (8) regular in a neighbourhood of $z = 0$ are constants.

Let us next consider the case when the functional equation (8) is to be satisfied by a Laurent series $f(z)$ with $I < 0$. Then

$$g(z) = f(z)^{-1},$$

is a power series such that

$$g(z^q)^{-1} = P_0 + P_1 g(z)^{-1} + \ldots + P_m g(z)^{-m},$$

or on multiplying by $g(z)^m g(z^q)$,

$$g(z)^m = P_0 g(z)^m g(z^q) + P_1 g(z)^{m-1} g(z^q) + \ldots + P_m g(z^q). \tag{10}$$

To this functional equation corresponds the polynomial

$$Q(u, v) = u^m - v(P_0 u^m + P_1 u^{m-1} + \ldots + P_m),$$

and the algebraic function $u = U(v)$ defined by $Q(U(v), v) = 0$. Since by hypothesis P_m does not vanish, $U(v)$ has a branch of the form

$$U_0(v) = P_m^{-1/m} v^{1/m} + \text{terms in higher powers of } v.$$

It follows then from the condition (C$_3$) that necessarily

$$a = m = q \quad \text{and} \quad b = 1.$$

Hence the functional equation (8) has exactly then an analytic solution with a pole at $z = 0$ when $q = m$. The condition (C$_4$) can be satisfied by taking for the suffix r any multiple of m.

10.

The considerations that allowed us to establish theorem 1 do not only provide an existence proof, but also give a method for the actual computation and tabulation of the analytic solutions, say by means of a computer or a programmable calculator.

This requires, firstly, the evaluation of a set of coefficients

$$p_b, p_{b+1}, \ldots, p_{b+L},$$

and hence also of the derived coefficients

$$\pi_0 = 1, \pi_1, \ldots, \pi_L, \tag{11}$$

of the branch $U_0(v)$ of the algebraic function $U(v)$. This calculation can be carried out say, by means of Newton's classical polygon method.

Secondly, once the coefficients (11) have been found, the recursive formulae (E) enable us to determine also the set of coefficients

$$\phi_1, \phi_2, \ldots, \phi_L,$$

of $f(z)$. We so obtain a polynomial

$$f_0 + f_r z^r (1 + \phi_1 z + \phi_2 z^2 + \ldots + \phi_L z^L),$$

which, by the convergence of the power series for $f(z)$, approximates this function very well if L is sufficiently large and $|z|$ is sufficiently small.

Thirdly, the functional equation (F) permits then to compute $f(z)$ for larger and larger values of $|z|$ as long as z lies in the unit disc D. However, since (F) defines $f(z)$ as a multi-valued algebraic function of $f(z^q)$, at each step of the calculation the right root must be chosen. When $f(z)$ is single-valued in D, this can be done by continuity; otherwise special considerations are required.

For the functional equation

$$f(z)^2 - f(z^2) + c = 0,$$

where $c > 0$ is a real parameter, such an investigation has been carried out in some detail in my paper (Mahler 1981).

<div align="center">11.</div>

A simple transformation of the variable connects the functional equation

$$P(f(z), f(z^q)) = 0 \tag{F}$$

of theorem 1 with the difference equation

$$P(F(Z), F(Z+1)) = 0, \tag{D}$$

where $P(u, v)$ is the same polynomial as before.

For real components x, y, X, Y, write the previous complex variable z as $z = x + y\mathrm{i}$ while $Z = X + Y\mathrm{i}$ is a second complex variable. Here assume that

$$z = \mathrm{e}^{-q^Z} \tag{12}$$

where $q \geqslant 2$ is again an integer, and

$$q^Z = \mathrm{e}^{Z \ln q}.$$

Thus the relation between z and Z is given by

$$z = \mathrm{e}^{-q^X[\cos(Y \ln q) + \mathrm{i}\sin(Y \ln q)]}.$$

Denote by S the horizontal strip

$$S \colon |Y| < \pi/2 \ln q$$

in the Z-plane. In S,

$$\cos(Y \ln q) > 0,$$

so that for the corresponding point z

$$|z| = \mathrm{e}^{-q^X \cos(Y \ln q)} < 1.$$

Hence S is mapped by (12) into the set

$$D_0 \colon 0 < |z| < 1,$$

for the exponential function defining z cannot be zero.

Next denote by $\epsilon > 0$ an arbitrarily small constant, and now restrict Z to the narrower strip

$$S_\epsilon \colon |Y| \leqslant (\pi - \epsilon)/2 \ln q$$

contained in S. For points Z in S_ϵ it is clear that uniformly in Z,

$$|z| \to 0 \quad \text{as} \quad X \to +\infty,$$

and

$$|z| \to 1 \quad \text{as} \quad X \to -\infty.$$

If Z lies one on of the two lines

$$Y = \pm \pi/2 \ln q$$

that bound S, then

$$z = e^{\pm iq^X}$$

has absolute value 1, thus lies on the frontier of D.

By means of these properties, theorem 1 leads immediately to the following consequence.

THEOREM 2. *Let the notation be as in theorem* 1, *and let the four conditions* (C_1), (C_2), (C_3), (C_4) *be satisfied. Then*

$$P(F(Z), F(Z+1)) = 0 \tag{D}$$

has a formal solution

$$F(Z) = f_0 + f_r e^{-rq^Z} \left(1 + \sum_{j=1}^{\infty} \phi_j e^{-jq^Z}\right)$$

which is not a constant. Moreover, this series converges uniformly in all points of the strip S_ϵ for which X is sufficiently large, hence is here regular and analytic. By means of the functional equation (D), *the function $F(Z)$ can be continued into the whole strip S as an analytic function, but may have here an infinite sequence of poles and algebraic branch points tending to the frontier of S.*

Just as in §8, on replacing $F(Z)$ by $F(Z)^{-1}$, we can obtain similar results for the solutions

$$F(Z) = \sum_{j=I}^{\infty} f_j e^{-jq^Z},$$

where I is a negative integer. Such solutions tend thus rather rapidly to infinity as for $Z \in S_\epsilon$ the real part of Z tends to $+\infty$.

REFERENCES

Mahler, K. 1976 On a class of nonlinear functional equations connected with modular functions. *J. Aust. math. Soc.* A **22**, 65–118.

Mahler, K. 1981 On a special nonlinear functional equation. *Proc. R. Soc. Lond.* A **378**, 155–178.