BULL. AUSTRAL. MATH. SOC.
VOL. 29 (1984), 101-108.
10-02

SOME SUGGESTIONS FOR FURTHER RESEARCH

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At the age of 80 I cannot expect to do much more mathematics. I may, however, state a number of questions where perhaps further research might lead to interesting results.

1. A problem on Liouville numbers

Long ago, Maillet [9] proved that if X is a Liouville number, and $f(z)$ is a rational function with rational coefficients, then also $f(X)$ is a Liouville number. This leads me to ask the following question.

Which analytic functions $f(z)$ have the property that if X is any Liouville number, then so is $f(X)$? In particular, are there entire transcendental functions with this property?

The difficulty of this problem lies of course in the fact that the set of all Liouville numbers is non-enumerable.

2. Cantor's set

Cantor's set C consists of all numbers

$$X = \sum_{h=1}^{\infty} 2d_h 3^{-h}$$

where all the coefficients d_h are either 0 or 1, thus the products $2d_h$ are either 0 or 2. The related numbers

Received 27 July 1983.

$$x = \sum_{h=1}^{\infty} d_h 2^{-h}$$

form the closed interval $c = [0, 1]$. We write $X = M(x)$. Whenever X is rational, so is x , and *vice versa*.

If X and x are rational, the sequence $d_1, d_2, d_3, \ldots$ is periodic, say of period length P and with p terms before the start of the periodic part. Then two numerators N and n exist such that

$$X = N/3^p\left(3^P-1\right) \quad \text{and} \quad x = n/2^p\left(2^P-1\right) ,$$

where the numerators and the denominators need not be relatively prime. Here the difficult problem of the simultaneous factorisation of

$$3^P - 1 \quad \text{and} \quad 2^P - 1$$

for the same P plays a role if both X and x are to be written as reduced fractions. While in many examples the exact denominator of X is larger than that of x , there are also cases when the opposite is true.

Next let both X and x be irrational, $X = M(x)$. The best rational approximations of x are the convergents,

$$u_r/v_r = n_r/2^{p_r}\left(2^{P_r}-1\right) \quad \text{say,}$$

of the continued fraction of x . The images under M ,

$$U_r/V_r = M\left(u_r/v_r\right) = N_r/3^{p_r}\left(3^{P_r}-1\right) ,$$

are still good approximations of X , but need not be convergents of the continued fraction for X . Nor need the convergents of the continued fraction for X lie in Cantor's set. Thus the following problem arises.

How close can irrational elements of Cantor's set be approximated by rational numbers

(i) in Cantor's set, and

(ii) by rational numbers not in Cantor's set?

A second problem is much more difficult. It asks the following question.

Are irrational elements of Cantor's set necessarily *transcendental? Thus does Cantor's set contain no irrational algebraic elements?*

A possible approach to this question consists in the study of the non-homogeneous linear expressions

$$\left|3^{p_r+P_r}X - 3^{p_r}X - N_r\right| .$$

It may be that a p-adic form of Schmidt's theorem on the rational approximations of algebraic numbers [10] holds for such expressions.

3. A non-linear functional equation

Let $q \geq 2$ be an integer and $P(u, v)$ an irreducible polynomial with complex coefficients which contains both variables u and v to positive degrees. Recently, in a paper which is yet to appear, I gave necessary and sufficient conditions under which the functional equation

$$P\left(f(z), f\left(z^q\right)\right) = 0 \tag{P}$$

has a non-constant analytic solution $f(z)$ which is either regular in a neighbourhood of the point $z = 0$ or has a pole at this point. The functional equation (P) enables us to continue this function into the unit disk $|z| < 1$, but $f(z)$ need no longer be single-valued in this disk, but may have infinitely many algebraic branch points.

The following question is therefore appropriate.

Under which conditions on q *and* $P(u, v)$ *has the functional equation* (P) *a solution* $f(z)$ *which is regular and single valued in the region* $0 < |z| < 1$, *but which has an essential singularity at* $z = 0$?

I conjecture that a suitable change of variable will lead to a solution.

4. Fourier coefficients of modular forms

In my paper (Mahler [8]) I applied the transformation theory of modular functions to establish simple systems of recursive formulae for the Fourier coefficients of such functions, for example, for $j(\omega)$ and $j(\omega)^{1/3}$. The method was purely algebraic.

Can this method be generalised so as to lead to analogous recursive formulae for the Fourier coefficients of modular forms?

5. Functions algebraic at algebraic points

Let Σ and Σ_0 be the sets of all meromorphic transcendental, and of all rational, functions $f(z)$ with the following property.

If α is any algebraic number, then $f(z)$ has in a neighbourhood of $z = \alpha$ either a power series

$$f(z) = \sum_{h=0}^{\infty} f_h(z-\alpha)^h$$

or a Laurent series

$$f(z) = \sum_{h=-n}^{\infty} f_h(z-\alpha)^h$$

in which all the coefficients f_h are algebraic numbers.

It was a classical result by Faber ([3]; see also Mahler [7]) that the set Σ is not empty and even contains entire transcendental functions. It is further clear that if $R(z, w_1, \ldots, w_r)$ is a polynomial with algebraic coefficients and if $f_1(z), \ldots, f_r(z)$ are any functions in Σ, then

$$P(z, f_1(z), \ldots, f_r(z))$$

is an element of either Σ or of Σ_0; further, with every function in Σ also all its derivatives belong to Σ.

I propose the following problem.

Can a function $f(z)$ in Σ satisfy an algebraic differential equation?

6. Applications of ceilings

In my paper (Mahler [6]) I introduced the notion of ceilings in algebraic number fields and used it to establish general lower and upper bounds for all the valuations of all the basis elements of any fractional divisor or ideal of the field. I now suggest the following investigations.

Let K and L be two finite algebraic number fields over $\mathbb{Q}$ *where K is a proper subfield of L. Study the relations between the ceilings $\lambda(\underline{p})$ of K and the ceilings $\Lambda(\underline{P})$ of L, and deduce inequalities linking the bases of ideals $\underline{\underline{a}}$ in K with the bases of ideals $\underline{\underline{A}}$ in L. Apply the results to the study of relative algebraic number fields.*

Investigate the bases of the differents of K and L.

Use my general results on compound convex bodies (Mahler [4]) *and on invariant matrices* (Mahler [5]) *to study the ideal bases in fields* $\mathbb{Q}(\theta, \theta', \ldots, \theta^{(k-1)})$ *obtained by adjoining to the rational field more than one root of an irreducible polynomial in* $\mathbb{Q}[x]$.

7. On a recursively constructed set of points

In the paper (Billing and Mahler [1]) the following recursive construction of a set of points was given.

In the projective plane denote by $[k, k']$ the line through two distinct points k and k', and by (L, L') the point of intersection of two distinct lines L and L'.

Let now $0, 1, 2, 3$, and 4 be five distinct points no three of which are collinear. Define five further points $-1, \ldots, -5$ by

$$-3 = ([0, 3], [1, 2]) \ , \quad -4 = ([0, 4], [1, 3]) \ ,$$

$$-5 = ([1, 4], [2, 3]) \ , \quad -1 = ([0, 1], [-3, 4]) \ , \quad -2 = ([0, 2], [-1, 3]) \ ,$$

then for every integer $k \geq 5$ the point k as the intersection of all the lines

$$[k'-k, -k'] \quad \text{where} \quad 0 \leq k' < k/2 \ ,$$

and for every integer $k \geq 6$ the point $-k$ as the intersection of all the lines

$$[k-k', k'] \ , \text{ where } \quad 1 \leq k' < k/2 \ .$$

Then a sufficient condition for three points k, k', k'' to be collinear is that $k + k' + k'' = 0$, but this condition need not be necessary if the original points $0, 1, \ldots, 4$ are in special positions.

This construction leads to a sequence of points k on a cubic, which may possibly be degenerate. If it is of genus 1, then the coordinates of the point k take the form of elliptic functions $\phi_j(k\alpha)$, $j = 1, 2, 3$, all with the same periods; α is a certain number which is not a period. If no rational multiple of α is a period, then the points k are all distinct, and in the case of a real cubic lie dense on its infinite branch.

Can an analogous construction (which, however, may be algebraic of higher order) *be given for sets of* p *points on algebraic curves of genus* $p > 1$ *? Can something similar be done, say for Kummer's surface which can be defined by hyperelliptic functions of genus* 2 *?*

8. Second order Farey polynomials

In the paper (Brown and Mahler [2]) the following generalisation of the well known Farey sequences was introduced.

Let n be a positive integer, G_n the set of all polynomials

$$ax^2 + bx + c = [a, b, c]$$

with integral coefficients a, b, c satisfying

$$0 \le a \le n , \quad -n \le b \le n , \quad -n \le c \le n \quad (\text{when } a = 0 \text{, then } b \ge 1)$$

and let F_n be the subset of G_n which consists of all polynomials in G_n which have either two real irrational zeros, or are linear and so have just one rational zero. Form the set of all the zeros of all the polynomials in F_n and number these in order of increasing size; apply the same ordering also to the polynomials in F_n so that the quadratic polynomials occur twice and the linear ones only once. In the lowest case $n = 1$ we so so obtain the sequence

[1, 1, -1], [0, 1, 1], [1, -1, -1], [0, 1, 0],
[1, 1, -1], [0, 1, -1], [1, -1, -1]

with the corresponding zeros

$$-1.618 \ldots, -1, -0.618 \ldots, 0, 0.618 \ldots, 1, 1.618 \ldots .$$

F_2 contains 31 polynomials, F_3 contains 103, F_4 contains 223, F_5 contains 483, F_6 contains 763, F_7 contains 1311 polynomials, and so on.

Associate with each triplet of three consecutive polynomials in F_n their determinant. Then the original calculations suggested that, if the polynomial in the middle was not linear, then this determinant has one of the three values -1, 0 , or +1 ; other values could however be obtained when the polynomial in the middle was linear.

When Brown's calculations were extended here at Canberra, it turned out that already for $n = 7$ the determinant could have values distinct from -1, 0 , and +1 even when the middle polynomial was quadratic; the following exceptions were obtained:

$$[1, 6, 7], [1, 3, -6], [2, 7, -7] \text{ , determinant } -2 \text{ ,}$$

$$[7, -7, -2], [6, -3, -1], [7, 6, 1] \text{ , determinant } -2 \text{ ,}$$

and once more the same triplets of polynomials in reversed order for their second zeros.

The number of exceptional triplets increases with n , but I do not know how quickly.

Let us therefore propose the following investigations.

Denote by

$$x \to \frac{ax+b}{cx+d} \quad (ad - bc = \pm 1)$$

a fixed modular transformation, and let n *be very large. Are there sections of* F_n *which are transformed into each other by such a transformation?* (This is certainly so for $x \to -x$ and $x \to -x^{-1}$). *How are the elements of* F_{n+1} *not in* F_n *distributed amongst the elements of* F_n *?*

Study the general law of the determinants of triplets of consecutive polynomials in F_n .

If $-1 \leq X \leq +1$, *how close can* X *be approximated by elements of* F_n *?*

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