(5.4) $$a_0 - ia_2 = c, \quad a_1 = b(1-i), \quad a_3 = b(1+i),$$

where $b \in Z$. Further, (2.13) gives

(5.5) $$0 = a_0 \bar{a}_2 + a_1 \bar{a}_3 + a_2 \bar{a}_4 + a_3 \bar{a}_5 = a_0 \bar{a}_2 - \bar{a}_0 a_2 - 4ib^2,$$

so that

(5.6) $$q = |a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2 = 4b^2 + |a_0|^2 + |a_2|^2 = c^2 + 8b^2,$$

since

$$c^2 = (a_0 - ia_2)(\bar{a}_0 + i\bar{a}_2) = |a_0|^2 + |a_2|^2 + i(a_0 \bar{a}_2 - \bar{a}_0 a_2) = |a_0|^2 + |a_2|^2 - 4b^2,$$

by (5.5).

Thus q, which initially appeared to be expressed as a sum of eight squares, turns out to be expressible as a real binary quadratic form. As an illustration, we have for $q = 17$,

$$a_0 = -1-2i, \quad a_1 = -1+i, \quad a_2 = -2-2i, \quad a_3 = -1-i,$$

giving $b = -1$, $c = -3$.

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On two analytic functions

by

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1. Denote by U: $|z| < 1$ the open unit disk in the complex z-plane, and by T an arbitrary closed subset of U. Next let $g \geq 2$ be a fixed integer, and let n run over all non-negative integers. Finally let

$$p(z) = p_0 + p_1 z + \ldots + p_d z^d,$$

where $d \geq 1$, be a polynomial with complex coefficients satisfying

$$p(0) = p_0 = 1 \quad \text{and} \quad p(1) = 0.$$

Hence $p(z)$ is divisible by $1-z$, say of the form

$$p(z) = (1-z)\,q(z),$$

where

$$q(z) = q_0 + q_1 z + \ldots + q_{d-1} z^{d-1}$$

is a second polynomial with complex coefficients such that

$$q(0) = q_0 = 1.$$

We shall use the notations

$$P = |p_0| + |p_1| + \ldots + |p_d| \quad \text{and} \quad Q = |q_0| + |q_1| + \ldots + |q_{d-1}|$$

for the sums of the absolute values of the coefficients of $p(z)$ and $q(z)$, respectively.

It is then obvious that

$$|p(z) - 1| \leq P - 1 \quad \text{and} \quad |q(z)| \leq Q \quad \text{for} \quad z \in U.$$

In these inequalities z may be replaced by z^{g^n} since with z also z^{g^n} belongs to the disk U. In fact, the following stronger inequality

$$|p(z^{g^n}) - 1| \leq (P-1)\,|z|^{g^n}$$

holds if $z \in U$, and n is any non-negative integer.

2. The power series

$$\sum_{n=0}^{\infty} z^{g^n}$$

converges absolutely for $z \in U$, and it converges uniformly in z for $z \in T$. This implies that the infinite product

$$f(z) = \prod_{n=0}^{\infty} p(z^{g^n})$$

likewise converges absolutely for $z \in U$ and uniformly in z for $z \in T$. Therefore the function $f(z)$ is analytic and regular at all points of U and hence can on this disk be written as a convergent power series

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \quad \text{where} \quad f_0 = 1.$$

We shall later decide whether this function can be continued into a larger region.

3. We first study the behaviour of $f(z)$ on the positive real axis as z tends to 1. Denote by Z a real variable such that

$$z = e^{-Z} \quad \text{where} \quad 0 < Z \leqslant 1 \text{ and } Z \to 0,$$

and associate with Z the integer

$$N = [\{\log(1/Z)\}/\{\log g\}].$$

Here $[x]$ denotes as usual the integral part of x. As Z tends to zero, N tends to infinity.

Now put

$$f_1(z) = \prod_{n=0}^{N-1} p(z^{g^n}) \quad \text{and} \quad f_2(z) = \prod_{n=N}^{\infty} p(z^{g^n}),$$

so that

$$f(z) = f_1(z) f_2(z).$$

Here by the factorisation of $p(z)$, and by the relation between z and Z,

$$f_1(z) = \prod_{n=0}^{N-1} \{(1-e^{-g^n Z}) q(e^{-g^n Z})\}.$$

Further for $0 < Z \leqslant 1$,

$$e^{-g^n Z} \geqslant 1 - g^n Z, \quad \text{hence} \quad 0 < 1 - e^{-g^n Z} \leqslant 1-(1-g^n Z) = g^n Z,$$

so that

$$0 < \prod_{n=0}^{N-1} (1-e^{-g^n Z}) \leqslant \prod_{n=0}^{N-1} (g^n Z) = g^{(N-1)N/2} Z^N < g^{N^2/2} Z^N.$$

For the product of the factors q we use the trivial estimate

$$\prod_{n=0}^{N-1} |q(e^{-g^n Z})| \leqslant Q^N.$$

It follows that

$$|f_1(z)| < g^{N^2/2} Z^N Q^N.$$

Here by the definition of N,

$$g^N \leqslant \exp\{(\log g)\{\log(1/Z)\}/(\log g)\} = 1/Z.$$

Therefore finally,

$$|f_1(z)| < (Q^2 Z)^{N/2} \quad \text{for} \quad 0 < Z \leqslant 1.$$

4. As a partial product of the convergent product $f(z)$ also $f_2(z)$ converges for $z \in U$. An upper estimate for $f_2(z)$ as function of Z is obtained as follows.

From the upper estimate for $|p(z)-1|$,

$$|f_2(z)| = \prod_{n=N}^{\infty} |p(e^{-g^n Z})| \leqslant \prod_{n=N}^{\infty} (1+(P-1)e^{-g^n Z}) = \prod_{m=0}^{\infty} (1+(P-1)e^{-g^N g^m Z}).$$

Here

$$g^N Z \leqslant 1.$$

It follows then that

$$|f_2(z)| \leqslant \prod_{m=0}^{\infty} (1+(P-1)e^{-g^n})$$

where the infinite product on the right-hand side does not depend on z or Z and is convergent. Its value is a certain positive constant R, and therefore

$$|f_2(z)| \leqslant R \quad \text{for} \quad 0 < Z \leqslant 1.$$

On combining this estimate with that for $f_1(z)$, we arrive at the final result that

$$|f(z)| < (Q^2 Z)^{N/2} R \quad \text{for} \quad 0 < Z \leqslant 1.$$

By the relation between Z and N it implies the following result.

THEOREM 1. *Write $z = e^{-Z}$ and allow Z to tend to 0 along the positive real axis $0 < Z \leqslant 1$. Then, if $c > 0$ is an arbitrarily large constant, there exists* *a second constant $C > 0$ such that*

$$|f(z)| \leqslant C \cdot Z^c \quad \textit{as} \quad Z \to 0.$$

5. By definition,

$$f(z) = \prod_{n=0}^{\infty} p(z^{g^n})$$

and therefore

$$f(z) = p(z) f(z^g).$$

It follows that for all positive integers n,

(1) $$f(z) = f(z^{g^n}) \prod_{h=0}^{n-1} p(z^{g^h}).$$

Denote by U_0: $|z| = 1$ the unit circle which is the frontier of the unit disk U. Let further E be the set of all g^nth roots of unity ε, where n runs over the positive integers; this set E is everywhere dense on U_0.

Theorem 1 implies that if z tends to 1 along the positive real axis, then $f(z)$ tends to 0. Hence it follows from the functional equation (1) that more generally $f(z)$ tends to 0 if z tends to any element ε of E along the radius of U_0 from $z = 0$ to $z = \varepsilon$. This property of $f(z)$ allows to deduce that this function cannot be regular at any point of U_0. For otherwise $f(z)$ would also be regular on a whole sufficiently small arc A of U_0. But this arc A contains a dense set of points $z = \varepsilon$ of U_0, and at all these points $f(z)$ would have the value 0. Hence $f(z)$ would necessarily be identically equal to 0, contrary to $f(0) = 1$.

The following result has thus been established.

THEOREM 2. *The unit circle U_0 is the natural boundary of the function $f(z)$.*

6. Now denote by a an arbitrary positive parameter and by $s = \sigma + ti$, where σ and t are real numbers, a second complex variable. Associate with the power series

$$f(z) = \sum_{n=0}^{\infty} f_n z^n$$

the formal Dirichlet series

$$\varphi(s|a) = \sum_{n=0}^{\infty} f_n (a+n)^{-s}$$

with the same coefficients f_n. However, it remains uncertain under which conditions on the polynomial $p(z)$ this Dirichlet series has a region of convergence. We shall therefore define $\varphi(s|a)$ by a definite integral for which the convergence can be established.

For this purpose we apply Euler's integral for the gamma function in the form

$$\Gamma(s)(a+n)^{-s} = \int_0^{\infty} e^{-(a+n)Z} Z^{s-1}\, dZ.$$

As is well known, this integral converges if $a+n$ is real and positive and the real part σ of s is positive.

By a purely formal calculation,

$$\Gamma(s)\,\varphi(s|a) = \int_0^{\infty} e^{-aZ}\Big(\sum_{n=0}^{\infty} f_n e^{-nZ}\Big) Z^{s-1}\, dZ.$$

We therefore define from now on $\varphi(s|a)$ by the equation

(2) $$\Gamma(s)\,\varphi(s|a) = \int_0^{\infty} e^{-aZ} f(e^{-Z}) Z^{s-1}\, dZ,$$

which certainly converges for $a > 0$ and $\sigma > 0$.

The condition for a will be left unchanged, but it will now be shown that the restriction on s may be omitted.

It is clear that the integrand in (2) is regular for finite positive Z and that the integrability may be disturbed only at the two points $Z = 0$ and $Z = \infty$.

As Z tends to 0, the factor e^{-aZ} remains regular. Under the same assumption for Z, by Theorem 1,

$$|f(e^{-Z}) Z^{s-1}| \leqslant CZ^c \cdot Z^{\sigma-1},$$

where we may take for c so large a positive number that $c+\sigma-1 > 0$. Thus the integrand of (2) is integrable at $Z = 0$ since it tends to 0.

Finally, as Z tends to ∞, $f(e^{-Z})$ tends to $f(0) = 1$, while $e^{-aZ} Z^{s-1}$ tends to 0 for every value of s. Hence the integration to ∞ is valid.

Let now s be restricted to a bounded closed region in the complex s-plane, and let a be restricted to a finite interval on the real positive axis. Then the integration is uniform in both s and a. We obtain therefore the following result:

THEOREM 3. *Let a be a positive real parameter and s a complex variable. Then the function $\Gamma(s)\,\varphi(s|a)$ and hence also the function $\varphi(s|a)$ is entire in s and continuous in a.*

Here the gamma function has poles at all the non-positive integers. It follows therefore that the entire function $\varphi(s|a)$ has zeros at all the points $s = 0, -1, -2, \ldots$

7. In its dependence on a, the function $\varphi(s|a)$ satisfies a simple functional equation which can be derived from the functional equation for $f(z)$, as follows.

By definition,

$$f(z) = p(z)f(z^g), \quad \text{where} \quad p(z) = p_0+p_1z+\ldots+p_dz^d,$$

hence

$$f(e^{-Z}) = (p_0+p_1e^{-Z}+p_2e^{-2Z}+\ldots+p_de^{-dZ})f(e^{-dZ}).$$

Therefore

$$\Gamma(s)\varphi(s|a) = \int_0^\infty e^{-aZ}f(e^{-gZ})(p_0+p_1e^{-Z}+\ldots+p_de^{-dZ})Z^{s-1}\,dZ$$

$$= \sum_{h=0}^{d} p_h \int_0^\infty e^{-aZ}f(e^{-gZ})e^{-hZ}Z^{s-1}\,dZ.$$

Here replace gZ by the new variable ζ. Then this formula becomes

$$\Gamma(s)\varphi(s|a) = \sum_{h=0}^{d} p_h \int_0^\infty e^{-\frac{a+h}{g}\zeta}f(e^{-\zeta})(\zeta/g)^{s-1}\,d\zeta/g$$

$$= \sum_{h=0}^{d} p_h g^{-s}\Gamma(s)\varphi\left(s\left|\frac{a+h}{g}\right.\right).$$

Hence $\varphi(s|a)$ satisfies the functional equation

(3) $$\varphi(s|a) = \sum_{h=0}^{d} p_h g^{-s}\varphi\left(s\left|\frac{a+h}{g}\right.\right).$$

On differentiating the integral for $\Gamma(s)\varphi(s|a)$ partially with respect to a, we obtain the further identity

(4) $$\frac{\partial}{\partial a}\varphi(s|a) = -\varphi(s+1|a).$$

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On some estimates involving the number of prime divisors of an integer

by

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Dedicated to Professor Paul Erdős on the occasion of his 75th birthday

1. Introduction and statement of results. Let as usual $\Omega(n)$ and $\omega(n)$ denote the number of all prime factors of $n \geqslant 1$ and the number of distinct prime factors of n, respectively. Further let $P(n)$ denote the largest prime factor of $n \geqslant 2$, and let $P(1) = 1$. The functions $\Omega(n)$, $\omega(n)$ and $P(n)$ determine to a large extent the distribution of prime divisors of n. In many problems involving $P(n)$ one often encounters the function

(1.1) $$\psi(x, y) = \sum_{n \leqslant x, P(n) \leqslant y} 1,$$

which represents the number of positive integers $\leqslant x$ all of whose prime factors are $\leqslant y$. An extensive literature on $\psi(x, y)$ exists, and recently (see [7], [8]) important developments in this field have been made. The new results on $\psi(x, y)$ are likely to find many applications, and in [11] they were used to obtain information about local densities of a certain class of arithmetical functions over integers with small prime factors. Several results concerning the local behaviour of $\psi(x, y)$ were derived in [11], and some of these will be needed in the proof of

THEOREM 1. *Let $y \leqslant x$, $\log y/\log\log x \to \infty$ as $x \to \infty$, and let p denote prime numbers. Then we have uniformly*

(1.2) $$\sum_{n \leqslant x, P(n) \leqslant y} (\Omega(n)-\omega(n)) = \psi(x, y)\left(\sum_p \frac{1}{p^2-p} + O\left(\frac{\log\log x}{\log y}\right)\right).$$

Asymptotic estimates of sums involving $\Omega(n)$, $\omega(n)$ and reciprocals of $P(n)$ elucidate the distribution of prime factors of n, and they were studied in [5], [6], and [10]. In particular, it was proved in [6] that

(1.3) $$\sum_{n \leqslant x} \frac{\Omega(n)-\omega(n)}{P(n)} = \left\{c + O\left(\frac{(\log\log x)^{3/2}}{\log^{1/2} x}\right)\right\} \sum_{n \leqslant x} \frac{1}{P(n)}$$

holds for a suitable constant $c > 0$, and that, as $x \to \infty$,